

Fourier based statistics for irregular spaced spatial data

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Abstract

A class of Fourier based statistics for irregular spaced spatial data is introduced, examples include, the Whittle likelihood, a parametric estimator of the covariance function based on the ℓ_2 -contrast function and a simple nonparametric estimator of the spatial autocovariance which is a non-negative function. The Fourier based statistic is a quadratic form of a discrete Fourier-type transform of the spatial data. Evaluation of the statistic is computationally tractable, requiring $O(b^2)$ operations, where b are the number Fourier frequencies used in the definition of the statistic (which varies according to the application). The asymptotic sampling properties of the statistic are derived using mixed spatial asymptotics, where the number of locations grows at a faster rate than the size of the spatial domain and under the assumption that the spatial random field is stationary and the irregular design of the locations are independent, identically distributed random variables. The asymptotic analysis allows for both the case that the frequency domain over which the estimator is defined is fixed and the case that the frequency domain grows with the spatial domain.

Keywords and phrases: Fixed and increasing frequency domain asymptotics, mixed spatial asymptotics, random locations, spectral density function, stationary spatial random fields.

1 Introduction

In recent years irregular spaced spatial data has become ubiquitous in several disciplines as varied as the Geosciences to econometrics. The analysis of such data poses several challenges which do not arise in data which is sampled on a regular lattice. One important problem is the computational costs when dealing with large irregular sampled data sets. If spatial data is sampled on regular lattice then algorithms such as the Fast Fourier transform can be employed to reduce the computational burden (see, for example, Chen, Hurvich, and Lu (2006)). Unfortunately, such algorithms have little benefit if the spatial data is irregularly sampled. To address this issue, within the spatial domain, several authors, including, Vecchia (1988), Cressie and Huang (1999), Stein, Chi, and Welty (2004), have proposed estimation methods which are designed to reduce the computational burden.

In contrast to the above references, Fuentes (2007) argues that working within the frequency domain can simplify the problem. Fuentes assumes that the irregular spaced data can be embedded on a grid and the missing mechanism is deterministic and ‘locally smooth’. Based on these assumptions Fuentes proposes a tapered Whittle estimator to estimate the parameters of a spatial covariance function. However, a possible drawback, is that if the locations are extremely irregular, the local smooth assumption will not hold. Therefore, in order to work within the frequency domain, methodology and inference devised specifically for irregular sampled data is required. Matsuda and Yajima (2009) and Bandyopadhyay and Lahiri (2009) have pioneered this approach, by assuming that the irregular locations are independent, identically distributed random variables (thus allowing the data to be extremely irregular) and define the irregular sampled discrete Fourier transform (DFT) as

$$J_n(\boldsymbol{\omega}) = \frac{\lambda^{d/2}}{n} \sum_{j=1}^n Z(\mathbf{s}_j) \exp(i\mathbf{s}_j' \boldsymbol{\omega}), \quad (1)$$

where $\mathbf{s}_j \in [-\lambda/2, \lambda/2]^d$ denotes the spatial locations observed in the space $[-\lambda/2, \lambda/2]^d$ and $\{Z(\mathbf{s}_j)\}$ denotes the spatial random field at these locations. It’s worth mentioning a similar transformation on irregular sampled data goes back to Masry (1978), who defines the discrete Fourier transform of Poisson sampled continuous time series. Using the definition of the DFT given in (1) Matsuda and Yajima (2009) defines the Whittle likelihood for the spatial data as

$$\int_{\Omega} \left(\log f(\boldsymbol{\omega}; \theta) + \frac{|J_n(\boldsymbol{\omega})|^2}{f(\boldsymbol{\omega}; \theta)} \right) d\boldsymbol{\omega}, \quad (2)$$

where Ω is a compact set in \mathbb{R}^d , here we will assume that $\Omega = [-C, C]^d$ and $f(\boldsymbol{\omega}; \theta)$ is the spectral density function. A clear advantage of this approach is that it avoids the

inversion of a large matrix. However, one still needs to evaluate the integral, which can be computationally quite difficult, especially if the aim is to minimise over the entire parameter space of θ .

For frequency domain methods to be computationally tractable and to be used as a viable alternative to ‘spatial domain’ methods, the frequency domain needs to be gridified in such a way that tangible estimators are defined on the grid. The problem is how to choose an appropriate lattice on \mathbb{R}^d . A possible solution can be found in Bandyopadhyay and Subba Rao (2014), Theorem 2.1, who show that on the lattice $\{\boldsymbol{\omega}_{\mathbf{k}} = (\frac{2\pi k_1}{\lambda}, \dots, \frac{2\pi k_d}{\lambda}); \mathbf{k} \in \mathbb{Z}^d\}$ the DFT $\{J_n(\boldsymbol{\omega}_{\mathbf{k}})\}$ is ‘close to uncorrelated’ if the random field is second order stationary and the locations $\{\mathbf{s}_j\}$ are independent, identical, uniformly distributed random variables. Heuristically, this result suggests that $\{J_n(\boldsymbol{\omega}_{\mathbf{k}})\}$ contains all the information about the random field, such that one can perform the analysis on the transformed data $\{J_n(\boldsymbol{\omega}_{\mathbf{k}})\}$. For example, rather than use (2) one can use the discretized likelihood

$$\frac{1}{\lambda^d} \sum_{k_1, \dots, k_d = -C\lambda}^{C\lambda} \left(\log f(\boldsymbol{\omega}_{\mathbf{k}}; \theta) + \frac{|J_n(\boldsymbol{\omega}_{\mathbf{k}})|^2}{f(\boldsymbol{\omega}_{\mathbf{k}}; \theta)} \right), \quad (3)$$

to estimate the parameters. Indeed Matsuda and Yajima (2009), Remark 2, mention that in practice one should use the discretized likelihood to estimate the parameters, but, unfortunately, they did not derive any results for the discretised likelihood.

The Whittle example and discussion above, illustrates the need for a proper statistical analysis of frequency domain statistics for irregular spatial data. Therefore, in this paper we will study a general class of frequency domain estimators, which includes (3) as an example. More precisely, we consider statistics which have the form

$$Q_{a,\lambda}(g; \mathbf{r}) = \frac{1}{\lambda^d} \sum_{k_1, \dots, k_d = -a}^a g(\boldsymbol{\omega}_{\mathbf{k}}) J_n(\boldsymbol{\omega}_{\mathbf{k}}) \overline{J_n(\boldsymbol{\omega}_{\mathbf{k}+\mathbf{r}})} \quad \mathbf{r} \in \mathbb{Z}^d, \quad (4)$$

where $J_n(\boldsymbol{\omega}_{\mathbf{k}})$ is defined in (1) and $g : \mathbb{R}^d \rightarrow \mathbb{R}$ is a weight function which depends on the application. In Section 2 we give examples of estimators which have the form (4), including the discretized Whittle likelihood, a spectral density estimator and a non-negative definite, nonparametric, spatial covariance function estimator. In terms of computation, evaluation of $\{J_n(\boldsymbol{\omega}_{\mathbf{k}}); k_1, \dots, k_d = -a, \dots, a\}$ requires $O(a^{2d})$ operations (as far as are aware the FFT cannot be used to reduce the number of operations for irregular spaced data). However, once $\{J_n(\boldsymbol{\omega}_{\mathbf{k}}); k_1, \dots, k_d = -a, \dots, a\}$ has been evaluated the evaluation of $Q_{a,\lambda}(g; \mathbf{r})$ only requires $O(a)$ operations.

In this paper we derive the asymptotic sampling properties of $Q_{a,\lambda}(g; \mathbf{r})$ under the mixed asymptotic framework, introduced in Hall and Patil (1994). This is where $\lambda \rightarrow \infty$ (the size

of the spatial domain grows) and the number of observations $n \rightarrow \infty$ in such a way that $\lambda^d/n \rightarrow 0$. In other words, the sampling density gets more dense as the spatial domain grows. We should also mention that there exists other asymptotic set-ups for spatial statistics, including fixed asymptotics (where λ is kept fixed but the number of locations, n grows) considered in Stein (1994, 1999), however, within this framework the sampling properties of $Q_{a,\lambda}(g; \mathbf{r})$ would be different.

It is clear that $Q_{a,\lambda}(g; 0)$ closely resembles the integrated weighted periodogram estimators which are often used in time series (see for example, Walker (1964), Hannan (1971), Dunsmuir (1979), Dahlhaus and Janas (1996) Can, Mikosch, and Samorodnitsky (2010) and Niebuhr and Kreiss (2014)). However, there are some important difference, which means the analysis of (4) is very different to those in classical time series. In regular sampled time series (or spatial processes defined on a grid), the domain over which the periodogram can be defined is fixed, usually $[0, 2\pi]$, outside this domain aliasing occurs. Therefore the fundamental frequencies of the time series periodogram are defined over a increasingly dense, but fixed domain. Returning to irregular sampled spatial data, one can also fix the frequency domain of $Q_{a,\lambda}(g; \mathbf{r})$, which means that $a = C\lambda$, where C is a fixed constant. For example, it is clear that the Whittle likelihood defined in (2) can only be defined on a fixed frequency domain and is not well defined over the entire frequency domain \mathbb{R}^d . In Section 2 we give examples of statistics which have the form (4), where the frequency domain is fixed with $a = C\lambda$. However, in the case that spatial process is irregularly sampled, it is not necessary to constrain $Q_{a,\lambda}(g; \mathbf{r})$ to a fixed frequency domain. The nature of ‘truely’ irregular sampling means that, unlike regularly spaced or near regularly spaced data, the DFT can estimate high frequencies, without the curse of aliasing (a phenomena which was noticed as early as Shapiro and Silverman (1960) and Beutler (1970)).

The analysis of $Q_{a,\lambda}(g; \mathbf{r})$ in the case that the domain is kept fixed yields results which are analogous to those for regular sampled time series and spatial processes (and with a little work follow from Bandyopadhyay and Subba Rao (2014) Theorem 2.1). However, in the increasing frequency domain framework, the number of terms within the summand of $Q_{a,\lambda}(g; \mathbf{r})$ is a^d which is increasing at a rate faster than the standardisation λ^d . This turns the problem into one that is technically rather challenging. For example, the methods used to analysis fixed frequency estimators cannot be applied to increasing frequency estimators and new techniques are required. Therefore, one of the aims in this paper is to develop the machinery for the analysis of increasing frequency domain estimators.

To facilitate the analysis, in Section 3 we start with the special case that the locations $\{\mathbf{s}_j\}$

are uniformly distributed on $[-\lambda/2, \lambda/2]^d$. This motivates the methodology, and in Section 3 we obtain an asymptotic expression for the expectation and variance of $Q_{a,\lambda}(g; \mathbf{r})$ in the both the fixed and increasing frequency domain. In particular, we show that by imposing some mild restrictions on the rate of growth of a (with respect to λ) and some regularity conditions on the spectral density function of the spatial process, the sampling properties for both the fixed frequency and increasing frequency domains are unified. Furthermore, the asymptotic normality of $Q_{a,\lambda}(g; \mathbf{r})$ is derived for both the fixed and increasing frequency domain (which again requires some mild conditions on the rate of growth of the frequency domain). In Section 4 we consider the sampling properties of $Q_{a,\lambda}(g; \mathbf{r})$ under the assumption that the locations $\{\mathbf{s}_j\}$ are no longer uniformly distributed, and follow the general sampling scheme described in Matsuda and Yajima (2009) and Bandyopadhyay and Lahiri (2009). Under these conditions we show that the DFTs are no longer ‘near uncorrelated’ at the frequencies $\{\boldsymbol{\omega}_k\}$, nevertheless by using the machinery developed for the uniform case we show that similar sampling properties as those derived in the uniform case hold true for non-uniformly distributed locations. The proofs for the results in these sections can be found in Sections 5-7.

The technical proofs can be found in the Appendix.

2 Preliminary results and assumptions

We observe the spatial random field $\{Z(\mathbf{s}); \mathbf{s} \in \mathbb{R}^d\}$ at the locations $\{\mathbf{s}_j\}_{j=1}^n$. Throughout this paper we will use the following assumptions on the spatial random field.

Assumption 2.1 (Spatial random field) (i) $\{Z(\mathbf{s}); \mathbf{s} \in \mathbb{R}^d\}$ is a second order stationary random field with mean zero and covariance function $c(\mathbf{s}_1 - \mathbf{s}_2) = \text{cov}(Z(\mathbf{s}_1), Z(\mathbf{s}_2) | \mathbf{s}_1, \mathbf{s}_2)$.

We define the spectral density function as $f(\boldsymbol{\omega}) = \int_{\mathbb{R}^d} c(\mathbf{s}) \exp(-i\mathbf{s}'\boldsymbol{\omega}) d\mathbf{s}$.

(ii) $\{Z(\mathbf{s}); \mathbf{s} \in \mathbb{R}^d\}$ is a stationary Gaussian random field.

Let $\alpha(\omega) \leq C$ for $|\omega| \in [-1, 1]$ and $|\alpha(\omega)| \leq C|\omega|^{-2}$ for $|\omega| > 1$. It is worth mentioning, that to minimise notation we will often use $\sum_{\mathbf{k}=-a}^a$ to denote the multiple sum $\sum_{k_1=-a}^a \cdots \sum_{k_d=-a}^a$.

2.1 Properties of the DFTs

We first consider the behaviour of the DFTs under both uniform and non-uniform sampling of the locations.

Below we summarize Theorem 2.1, Bandyopadhyay and Subba Rao (2014), which defines frequencies where the Fourier transform is ‘close to uncorrelated’, this result requires the additional assumption on the distribution of the spatial locations.

Assumption 2.2 (Uniform sampling) *The locations $\{\mathbf{s}_j\}$ are independent uniformly distributed random variables on $[-\lambda/2, \lambda/2]^d$.*

Theorem 2.1 *Let us suppose that $\{Z(\mathbf{s}); \mathbf{s} \in \mathbb{R}^d\}$ is a stationary spatial random field whose covariance function (defined in Assumption 2.1(i)) satisfies $c(\mathbf{s}) \leq \prod_{j=1}^d \alpha(s_j)$. Furthermore, the locations $\{\mathbf{s}_j\}$ satisfy the Assumption 2.2. Then we have*

$$\begin{aligned} & \text{cov}[J_n(\boldsymbol{\omega}_{\mathbf{k}_1}), J_n(\boldsymbol{\omega}_{\mathbf{k}_2})] \\ = & \begin{cases} f(\boldsymbol{\omega}_{\mathbf{k}}) + O(\frac{1}{\lambda} + \frac{\lambda^d}{n}) & \mathbf{k}_1 = \mathbf{k}_2 (= \mathbf{k}) \\ O(\frac{1}{\lambda^{d-b}}) & \mathbf{k}_1 - \mathbf{k}_2 \neq 0 \text{ but } b \text{ elements of the} \\ & d\text{-dimensional vectors } \mathbf{k}_1 \text{ and } \mathbf{k}_2 \text{ are the same,} \end{cases} \end{aligned}$$

where $\boldsymbol{\omega}_{\mathbf{k}} = (\frac{2\pi k_1}{\lambda}, \dots, \frac{2\pi k_d}{\lambda})$ and $\mathbf{k} \in \mathbb{Z}^d$.

PROOF See Theorem 2.1, Bandyopadhyay and Subba Rao (2014). □

To understand what happens in the case that the locations are not uniformly sampled, we adopt the assumptions of Hall and Patil (1994), Matsuda and Yajima (2009) and Bandyopadhyay and Lahiri (2009) and assume that $\{\mathbf{s}_j\}$ are iid random variables with density $\frac{1}{\lambda^d} h(\frac{\cdot}{\lambda})$, where $h : [-1/2, 1/2]^d \rightarrow \mathbb{R}$. We use the following assumptions on the sampling density h .

Assumption 2.3 (Non-uniform sampling) *The locations $\{\mathbf{s}_j\}$ are independent distributed random variables on $[-\lambda/2, \lambda/2]^d$, where the density of $\{\mathbf{s}_j\}$ is $\frac{1}{\lambda^d} h(\frac{\cdot}{\lambda})$, and $h(\cdot)$ admits the Fourier representation*

$$h(\mathbf{u}) = \sum_{\mathbf{j} \in \mathbb{Z}^d} \gamma_{\mathbf{j}} \exp(i2\pi \mathbf{j}' \mathbf{u}).$$

- (i) *If no elements of $\mathbf{j} \in \mathbb{Z}^d$ are zero, then the Fourier coefficients satisfy $|\gamma_{\mathbf{j}}| \leq C \|\mathbf{j}\|_2^{-2}$ (where $\|\cdot\|_2$ denotes the Euclidean norm of a vector). Note that this assumption is satisfied if the second derivative of h is bounded on the d -dimensional torus $[-1/2, 1/2]^d$ (note that this condition can be induced on the data by tapering the data around the boundaries of the observed spatial field).*

- (ii) $\int_{[-1/2, 1/2]^d} h(\mathbf{u})^2 d\mathbf{u} = \sum_{\mathbf{j} \in \mathbb{Z}^d} |\gamma_{\mathbf{j}}|^2 = 1.$

We now show that under this general sampling scheme the near ‘uncorrelated’ property of the DFT given in Theorem 2.1 does not hold.

Theorem 2.2 *Let us suppose that $\{Z(\mathbf{s}); \mathbf{s} \in \mathbb{R}^d\}$ is a stationary spatial random field whose covariance function (defined in Assumption 2.1(i)) satisfies $c(\mathbf{s}) \leq \prod_{j=1}^d \alpha(s_j)$. Furthermore, the locations $\{\mathbf{s}_j\}$ satisfy Assumption 2.3. Then we have*

$$\text{cov}[J_n(\boldsymbol{\omega}_{\mathbf{k}_1}), J_n(\boldsymbol{\omega}_{\mathbf{k}_2})] = f(\boldsymbol{\omega}_{\mathbf{k}_2}) \sum_{\mathbf{j} \in \mathbb{Z}^d} \gamma_{\mathbf{j}} \gamma_{\mathbf{k}_2 - \mathbf{k}_1 - \mathbf{j}} + \frac{c(0) \gamma_{\mathbf{k}_2 - \mathbf{k}_1} \lambda^d}{n} + O\left(\frac{1}{\lambda}\right).$$

PROOF See Section 7. □

Thus we observe that in the case the locations are not sampled from a uniform distribution the DFTs are not (asymptotically) uncorrelated. However, they do satisfy the property

$$\text{cov}[J_T(\boldsymbol{\omega}_{\mathbf{k}_1}), J_T(\boldsymbol{\omega}_{\mathbf{k}_2})] = O\left(\left(1 + \frac{I(\mathbf{k}_1 \neq \mathbf{k}_2)}{\|\mathbf{k}_1 - \mathbf{k}_2\|_1}\right)\left[1 + \frac{\lambda^d}{n}\right] + \frac{1}{\lambda}\right).$$

In other words, the correlations between the DFTs decline the further apart the frequencies. A similar result was derived in Bandyopadhyay and Lahiri (2009) who show that the correlation between $J_n(\boldsymbol{\omega}_1)$ and $J_n(\boldsymbol{\omega}_2)$ are asymptotic uncorrelated if their frequencies are ‘asymptotically distant’ such that $\lambda^d \|\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2\|_1 \rightarrow \infty$.

2.2 Assumptions for the analysis of $Q_{a,\lambda}(\cdot)$

In order to asymptotically analysis the statistic, $Q_{a,\lambda}(g; \mathbf{r})$, we will assume that the spatial domain $[-\lambda/2, \lambda/2]^d$ grows as the number of observations $n \rightarrow \infty$. This means that as the sample size grows a large number of observations (those whose locations are far apart) will only be weakly correlated, consequently, the variance of $Q_{a,\lambda}(g; \mathbf{r})$ converges to zero as $\lambda \rightarrow \infty$. Furthermore, to ensure that $Q_{a,\lambda}(g; \mathbf{r})$ is asymptotically unbiased we will assume that the number of locations n increases at a faster rate than the spatial domain $[-\lambda/2, \lambda/2]^d$ expands, in other words $\lambda^d/n \rightarrow 0$. Together these conditions will ensure consistency of the estimator $Q_{a,\lambda}(g; \mathbf{r})$, and is the so called mixed asymptotics set-up proposed in Hall and Patil (1994), Hall, Fisher, and Hoffman (1994) and used in, for example, Lahiri (2003), Matsuda and Yajima (2009), Bandyopadhyay and Lahiri (2009), Bandyopadhyay, Lahiri, and Norman (2013), Bandyopadhyay, Lahiri, and Norman (2014) and Bandyopadhyay and Subba Rao (2014).

In the case that $a = C\lambda$ (where $|C| < \infty$), the asymptotic properties of $Q_{a,\lambda}(g; \mathbf{r})$ can be mostly derived by using either Theorems 2.1 or 2.2 (depending on the sampling of the

locations). For example, in the case that the locations are uniformly sampled, it follows from Theorem 2.1 that

$$\mathbb{E}[Q_{a,\lambda}(g; \mathbf{r})] = \begin{cases} \frac{1}{(2\pi)^d} \int_{2\pi[-C,C]^d} g(\boldsymbol{\omega}) f(\boldsymbol{\omega}) d\boldsymbol{\omega} + O(\frac{\lambda^d}{n} + \frac{1}{\lambda}) & \mathbf{r} = 0 \\ O(\frac{1}{\lambda}) & \text{otherwise} \end{cases}$$

with $\lambda^d/n \rightarrow 0$ as $\lambda \rightarrow \infty$ and $n \rightarrow \infty$ (the full details will be given in Lemma). Similarly, in the non-uniform case, by using Theorem 2.2 we have

$$\mathbb{E}[Q_{a,\lambda}(g; \mathbf{r})] = \frac{1}{(2\pi)^d} \sum_j \gamma_j \gamma_{\mathbf{r}-j} \int_{2\pi[-C,C]^d} g(\boldsymbol{\omega}) f(\boldsymbol{\omega}) d\boldsymbol{\omega} + O(\frac{\lambda^d}{n} + \frac{1}{\lambda}).$$

However, the above analysis only holds in the case that $a = O(\lambda)$, in other words the frequency domain is kept fixed as the spatial domain $[-\lambda/2, \lambda/2]^d$ grows. In the case that we allow the frequency domain to also grow then using Theorems 2.1 or 2.2 to analyse $\mathbb{E}[Q_{a,\lambda}(g; \mathbf{r})]$ will lead to errors of the order $O(\frac{a}{\lambda^2})$, which cannot be controlled without placing severe restrictions on expansion of the frequency domain. Therefore, one of the main contributions of this paper is to develop methods that deal with this situation, these methods will require the following assumptions on the weight function g and spectral density f .

Assumption 2.4 (i) Fixed Frequency domain *The frequency domain stays fixed as the spatial domain increases, such that $a = C\lambda$ where $C < \infty$. In addition, $\sup_{\boldsymbol{\omega} \in [-C,C]^d} |g(\boldsymbol{\omega})| < \infty$ and for all $1 \leq j \leq d$, $\sup_{\boldsymbol{\omega} \in [-C,C]^d} |\frac{\partial g(\boldsymbol{\omega})}{\partial \omega_j}| < \infty$.*

Assumptions on the spatial random field:

- (a) $c(\mathbf{s}) \leq \prod_{j=1}^d \alpha(u_j)$ (where α is defined in at the start of Section 2).
- (b) The spectral density function $f(\cdot)$ is Lipschitz continuous, ie for $1 \leq j \leq d$ and $\omega_{1j}, \omega_{2j} \in \mathbb{R}$ we have $|f(\omega_1, \dots, \omega_{j1}, \dots, \omega_d) - f(\omega_1, \dots, \omega_{j2}, \dots, \omega_d)| \leq C|\omega_{j1} - \omega_{j2}|$, for some finite constant C .

(ii) Increasing Frequency domain *The frequency domain grows as the spatial domain grows, ie. $a/\lambda \rightarrow \infty$ as $\lambda \rightarrow \infty$. In addition, the function g is bounded $\sup_{\boldsymbol{\omega} \in \mathbb{R}^d} |g(\boldsymbol{\omega})| < \infty$ and for all $1 \leq j \leq d$, $\sup_{\boldsymbol{\omega} \in \mathbb{R}^d} |\frac{\partial g(\boldsymbol{\omega})}{\partial \omega_j}| < \infty$.*

Assumptions on the spatial random field:

- (a) $f(\boldsymbol{\omega})$ is bounded, $\int_{\mathbb{R}^d} f(\boldsymbol{\omega}) d\boldsymbol{\omega} < \infty$ and $\int_{\mathbb{R}^d} f^2(\boldsymbol{\omega}) d\boldsymbol{\omega} < \infty$.

(b) There exists a $\beta_j : \mathbb{R}^d \rightarrow \mathbb{R}$ such that for all $1 \leq j \leq d$ the partial derivatives satisfy $|\frac{\partial f(\boldsymbol{\omega})}{\partial \omega_j}| \leq \beta_j(\boldsymbol{\omega})$, where $\beta_j(\cdot)$ is bounded and monotonically decreasing at the j th variable (ie. for $0 < \omega_{j1} < \omega_{j2}$, $\beta_j(\omega_1, \dots, \omega_{j1}, \dots, \omega_d) < \beta_j(\omega_1, \dots, \omega_{j2}, \dots, \omega_d)$) and $\int_{\mathbb{R}^{d-1}} \int_0^b \beta_j(\omega_1, \dots, \omega_j, \dots, \omega_d) d\omega_j d\omega_1 \dots d\omega_d \leq \Gamma(b)$, where $\Gamma : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function which is monotonically increasing but is not necessarily bounded.

Remark 2.1 Assumption 2.4(ii)(a,b) appears quite technical, but it is satisfied by a wide range of spatial covariance functions. For example, we now show that the exponential covariance (which belongs to the Matérn class), defined by $c(\|\mathbf{s}\|_2) = \phi \exp(-\|\mathbf{s}\|_2)$ (where $\|\cdot\|_2$ denotes the Euclidean distance) satisfies these assumptions. To see why, we consider the case $d = 1$ and $d = 2$. For $d = 1$ the spectral density function of the exponential covariance is

$$f(\omega) = \frac{\phi}{1 + \omega^2},$$

whereas for $d = 2$ the exponential covariance has the spectral density

$$f(\omega_1, \omega_2) = \frac{2\pi\phi}{(1 + \omega_1^2 + \omega_2^2)^{3/2}}.$$

It is straightforward to show that these spectral density functions satisfy Assumption 2.4(ii)(a,b).

A slight variant on the statistic $Q_{a,\lambda}(g; \mathbf{r})$ is to remove the ‘ridge’ term and define the statistic

$$\tilde{Q}_{a,\lambda}(g; \mathbf{r}) = Q_{a,\lambda}(g; \mathbf{r}) - \frac{1}{n} \sum_{\mathbf{k}=-a}^a g(\boldsymbol{\omega}_{\mathbf{k}}) \frac{1}{n} \sum_{j=1}^n Z(\mathbf{s}_j)^2 \exp(-i\mathbf{s}'\boldsymbol{\omega}_{\mathbf{r}}). \quad (5)$$

This statistic estimates the same quantity as $Q_{a,\lambda}(g; \mathbf{r})$, but does not require the assumption that $a^d = O(n)$. Therefore, for most of the paper our focus will be mainly on (5), noting that similar results can also be derived for $Q_{a,\lambda}(g; \mathbf{r})$.

Below we give some examples of statistics of the form (4) and (5) which satisfy Assumption 2.4.

Example 2.1 (Fixed frequency domain) (i) The discretized Whittle likelihood given in equation (3).

(ii) A nonparametric estimator of the spectral density function $f(\boldsymbol{\omega})$ is

$$\hat{f}_{\lambda,n}(\boldsymbol{\omega}) = \sum_{\mathbf{k}} W_b(\boldsymbol{\omega} - \boldsymbol{\omega}_{\mathbf{k}}) |J_n(\boldsymbol{\omega}_{\mathbf{k}})|^2 - \frac{1}{n} \sum_{\mathbf{k}=-a}^a W_b(\boldsymbol{\omega} - \boldsymbol{\omega}_{\mathbf{k}}) \frac{1}{n} \sum_{j=1}^n Z(\mathbf{s}_j)^2$$

where $W_b(\boldsymbol{\omega}) = b^{-d} \prod_{j=1}^d W(\frac{\omega_j}{b})$ and $W : [-1/2, 1/2] \rightarrow \mathbb{R}$ is a spectral window.

It is worth mentioning that the spectral density estimator given in Example 2.1(i) does not exactly satisfy the Assumptions 2.4(i)(a,b) (since $\sup_{\omega} |W_b(\omega)| = O(b^d)$), however it is straightforward to adapt the sampling properties derived in the later sections to this case.

Example 2.2 (Increasing frequency domain) (i) In Bandyopadhyay and Subba Rao (2014) a test statistic of the type

$$\tilde{a}_n(g; \mathbf{r}) = \tilde{Q}_n(g; \mathbf{r}) \quad \text{where } \mathbf{r} \in \mathbb{Z}^d / \{0\},$$

is proposed. We will show in Section 3 that the above converges to zero in probability as $\lambda \rightarrow \infty$, whereas if the spatial process is nonstationary, this property does not hold (see Bandyopadhyay and Subba Rao (2014) for the details). Using these differing behaviours Bandyopadhyay and Subba Rao (2014) test for stationarity.

(ii) A nonparametric estimator of the spatial (stationary) covariance function is

$$\hat{c}_n(\mathbf{v}) = \left(\frac{1}{\lambda^d} \sum_{\mathbf{k}=-a}^a |J_n(\omega_{\mathbf{k}})|^2 \exp(i\mathbf{v}'\omega_{\mathbf{k}}) \right) T\left(\frac{\mathbf{v}}{\lambda}\right) \quad (6)$$

where $T(\mathbf{u}) = \prod_{j=1}^d t(u_j)$ and $t : \mathbb{R} \rightarrow \mathbb{R}$ is the triangular kernel defined as $t(u) = (1 - |u|)$ for $|u| \leq 1$ and $t(u) = 0$ for $|u| > 1$. We observe that the Fourier transform of $\hat{c}_n(\mathbf{v})$ is

$$\hat{f}_\lambda(\omega) = \int_{\mathbb{R}^d} \hat{c}_n(\mathbf{v}) \exp(-i\omega'\mathbf{v}) d\mathbf{v} = \sum_{\mathbf{k}=-a}^a |J_n(\omega_{\mathbf{k}})|^2 \text{Sinc}^2[\lambda(\omega_{\mathbf{k}} - \omega)]$$

where $\text{Sinc}(\lambda\omega) = \prod_{j=1}^d \text{sinc}(\lambda\omega_j)$ and $\text{sinc}(\omega) = \sin(\omega)/\omega$. Clearly, $\hat{f}_\lambda(\omega)$ is non-negative, therefore, the sample covariance function $\{\hat{c}_n(\mathbf{v})\}$ is a non-negative definite function. Comparing this estimator to the three stage estimator proposed in Hall et al. (1994) we note that (6) is a lot simpler to compute. In the Hall et al. (1994) estimation scheme, first a nonparametric pre-estimator of the covariance function is evaluated, then the Fourier transform of the pre-estimator covariance function is taken, in the final stage, negative values of the Fourier transform are placed to zero and it's inverse Fourier transform evaluated.

(iii) The Whittle likelihood defined in (2) has the disadvantage that it can only be defined over a compactly supported frequency domain (which means that very high frequency information about the covariance function cannot be used in the estimation scheme).

However, the Whittle likelihood can be viewed as the entropy loss function, and there is no reason that other loss functions could not be used. For example, the ℓ_2 -loss function

$$\mathcal{L}_n(\theta) = \frac{1}{\lambda^d} \sum_{\mathbf{k}=-a}^a ||J_n(\boldsymbol{\omega}_{\mathbf{k}})|^2 - f(\boldsymbol{\omega}_{\mathbf{k}}; \theta)|^2,$$

does not require any restrictions on the support of the frequency domain. We observe that $\mathcal{L}_n(\theta)$ does not have the form given in (4), however, the asymptotic sampling properties of it's estimator are determined by it's first derivatives (with respect to θ)

$$\nabla_{\theta} \mathcal{L}_n(\theta) = \frac{-2}{\lambda^d} \sum_{\mathbf{k}=-a}^a \Re \nabla_{\theta} f(\boldsymbol{\omega}_{\mathbf{k}}; \theta) \{ |J_n(\boldsymbol{\omega}_{\mathbf{k}})|^2 - f(\boldsymbol{\omega}_{\mathbf{k}}; \theta) \},$$

which can be written in the form given in (4)

3 Sampling properties of $Q_{a,\lambda}(g; \mathbf{r})$ under uniform sampling

In this section we will derive the asymptotic sampling properties of $Q_{a,\lambda}(g; \mathbf{r})$ under the assumption that the locations are uniformly distributed. As we mentioned in the introduction the challenge is when the number of terms in the sum $Q_{a,\lambda}(g; \mathbf{r})$ grows at a faster rate than the standardisation λ^d . In this case, the analysis of $Q_{a,\lambda}(g; \mathbf{r})$ requires more delicate techniques than those used to prove Theorems 2.1 and 2.2. To give a flavour of our approach we outline the analysis of the expectation of $Q_{a,\lambda}(g; \mathbf{r})$. By writing $Q_{a,\lambda}(g; \mathbf{r})$ as a quadratic form it is straightforward to show that

$$\begin{aligned} & \mathbb{E} [Q_{a,\lambda}(g; \mathbf{r})] \\ &= C_2 \sum_{\mathbf{k}=-a}^a g(\boldsymbol{\omega}_{\mathbf{k}}) \frac{1}{\lambda^d} \int_{[-\lambda/2, \lambda/2]^d} c(\mathbf{s}_1 - \mathbf{s}_2) \exp(i\boldsymbol{\omega}'_{\mathbf{k}}(\mathbf{s}_1 - \mathbf{s}_2) - i\mathbf{s}'_2 \boldsymbol{\omega}_{\mathbf{r}}) d\mathbf{s}_1 d\mathbf{s}_2 + V_{\mathbf{r}}, \quad (7) \end{aligned}$$

where $C_2 = n(n-1)/n^2$ and $V_{\mathbf{r}} = \frac{c(0)I(\mathbf{r}=0)}{n} \sum_{\mathbf{k}=-a}^a g(\boldsymbol{\omega}_{\mathbf{k}})$. The proof of Theorem 2.1 is based on making a change of variables $\mathbf{v} = \mathbf{s}_1 - \mathbf{s}_2$ and then systematically changing the limits of the integral. However, applying this brute force method to $\mathbb{E} [Q_{a,\lambda}(g; \mathbf{r})]$ has the disadvantage that it aggregates the errors within the sum of $\mathbb{E} [Q_{a,\lambda}(g; \mathbf{r})]$. Instead, to further the analysis, we replace $c(\mathbf{s}_1 - \mathbf{s}_2)$ by it's Fourier transform $c(\mathbf{s}_1 - \mathbf{s}_2) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(\boldsymbol{\omega}) \exp(i\boldsymbol{\omega}'(\mathbf{s}_1 - \mathbf{s}_2)) d\boldsymbol{\omega}$. This reduces the first term in $\mathbb{E} [Q_{a,\lambda}(g; \mathbf{r})]$ to the Fourier transforms of step functions, which is the product of sinc functions, $\text{Sinc}(\frac{\lambda}{2}\boldsymbol{\omega})$ (noting that the Sinc function is defined in Example

2.2). Specifically, we obtain

$$\begin{aligned} \mathbb{E}[Q_{a,\lambda}(g; \mathbf{r})] &= C_2 \sum_{\mathbf{k}=-a}^a g(\boldsymbol{\omega}_{\mathbf{k}}) \int_{\mathbb{R}^d} f(\boldsymbol{\omega}) \text{Sinc}\left(\frac{\lambda \boldsymbol{\omega}}{2} + \mathbf{k}\pi\right) \text{Sinc}\left(\frac{\lambda \boldsymbol{\omega}}{2} + (\mathbf{k} + \mathbf{r})\pi\right) d\boldsymbol{\omega} + V_{\mathbf{r}} \\ &= C_2 \int_{\mathbb{R}^d} \text{Sinc}(\mathbf{y}) \text{Sinc}(\mathbf{y} + \mathbf{r}\pi) \left[\frac{1}{\lambda^d} \sum_{\mathbf{k}=-a}^a g(\boldsymbol{\omega}_{\mathbf{k}}) f\left(\frac{2\mathbf{y}}{\lambda} - \boldsymbol{\omega}_{\mathbf{k}}\right) \right] d\mathbf{y} + V_{\mathbf{r}}, \end{aligned}$$

where the last line above is due to a change of variables $\mathbf{y} = \frac{\lambda \boldsymbol{\omega}}{2} + \mathbf{k}\pi$. Since the spectral density function is absolutely integrable it is clear that $\left[\frac{1}{\lambda^d} \sum_{\mathbf{k}=-a}^a g(\boldsymbol{\omega}_{\mathbf{k}}) f\left(\frac{2\mathbf{y}}{\lambda} - \boldsymbol{\omega}_{\mathbf{k}}\right) \right]$ is uniformly bounded over \mathbf{y} and that $\mathbb{E}[Q_{a,\lambda}(g; \mathbf{r})] - V_{\mathbf{r}}$ is finite for all λ . Next, by making a series of approximations, we can obtain a well defined limit of $\mathbb{E}[Q_{a,\lambda}(g; \mathbf{r})]$. If $f\left(\frac{2\mathbf{y}}{\lambda} - \boldsymbol{\omega}_{\mathbf{k}}\right)$ were replaced with $f(-\boldsymbol{\omega}_{\mathbf{k}})$, then what remains in the integral are two shifted Sinc functions, which is zero if $\mathbf{r} \in \mathbb{Z}^d / \{0\}$. In Theorem 3.1 we show that under certain conditions on f and the rate of growth of a with respect to λ that this difference is asymptotically negligible.

Theorem 3.1 *Suppose Assumptions 2.1(i), 2.2 and*

(i) Assumption 2.4(i)(a,b) hold. Then we have

$$\begin{aligned} &\mathbb{E} \left[\tilde{Q}_{a,\lambda}(g; \mathbf{r}) \right] \\ &= \begin{cases} O\left(\frac{1}{\lambda^{d-b}}\right) & \mathbf{r} \in \mathbb{Z}^d / \{0\} \text{ and } b \text{ elements of } \mathbf{r} \text{ are zero} \\ \frac{1}{(2\pi)^d} \int_{\boldsymbol{\omega} \in 2\pi[-C,C]^d} f(\boldsymbol{\omega}) g(\boldsymbol{\omega}) d\boldsymbol{\omega} + O\left(\frac{1}{\lambda}\right) & \mathbf{r} = 0 \end{cases} \quad (8) \end{aligned}$$

and

$$\begin{aligned} &\mathbb{E} [Q_{a,\lambda}(g; \mathbf{r})] \\ &= \begin{cases} O\left(\frac{1}{\lambda^{d-b}}\right) & \mathbf{r} \in \mathbb{Z}^d / \{0\} \text{ and } b \text{ elements of } \mathbf{r} \text{ are zero} \\ \frac{1}{(2\pi)^d} \int_{\boldsymbol{\omega} \in 2\pi[-C,C]^d} f(\boldsymbol{\omega}) g(\boldsymbol{\omega}) d\boldsymbol{\omega} + O\left(\frac{1}{\lambda} + \frac{\lambda^d}{n}\right) & \mathbf{r} = 0 \end{cases} \quad (9) \end{aligned}$$

(ii) Suppose the Assumption 2.4(ii)(a) holds, then $\sup_a \left| \mathbb{E} \left[\tilde{Q}_{a,\lambda}(g; \mathbf{r}) \right] \right| < \infty$. However, under Assumption 2.4(ii)(a,b), then

$$\begin{aligned} &\mathbb{E} \left[\tilde{Q}_{a,\lambda}(g; \mathbf{r}) \right] \\ &= \begin{cases} O\left(\Gamma(a/\lambda) \frac{\log \lambda + \log a}{\lambda} + \frac{1}{n}\right) & \mathbf{r} \in \mathbb{Z}^d / \{0\} \\ \frac{1}{(2\pi)^d} \int_{\boldsymbol{\omega} \in \mathbb{R}^d} f(\boldsymbol{\omega}) g(\boldsymbol{\omega}) d\boldsymbol{\omega} + O\left(\Gamma(a/\lambda) \frac{\log \lambda + \log a}{\lambda} + \frac{1}{n}\right) & \mathbf{r} = 0 \end{cases} \quad (10) \end{aligned}$$

and

$$\begin{aligned} &\mathbb{E} [Q_{a,\lambda}(g; \mathbf{r})] \\ &= \begin{cases} O\left(\Gamma(a/\lambda) \frac{\log \lambda + \log a}{\lambda} + \frac{1}{n}\right) & \mathbf{r} \in \mathbb{Z}^d / \{0\} \\ \frac{1}{(2\pi)^d} \int_{\boldsymbol{\omega} \in \mathbb{R}^d} f(\boldsymbol{\omega}) g(\boldsymbol{\omega}) d\boldsymbol{\omega} + \frac{c(0)}{n} \sum_{\mathbf{k}=-a}^a g(\boldsymbol{\omega}_{\mathbf{k}}) + O\left(\Gamma(a/\lambda) \frac{\log \lambda + \log a}{\lambda} + \frac{\lambda^d}{n}\right) & \mathbf{r} = 0 \end{cases} \quad (11) \end{aligned}$$

PROOF See Section 5. □

From Theorem 3.1 we observe that asymptotically $Q_{a,\lambda}(g; \mathbf{r})$ and $\tilde{Q}_{a,\lambda}(g; \mathbf{r})$ are estimating the same quantity (up to a ‘ridge’ constant, $\frac{c(0)}{n} \sum_{\mathbf{k}=-a}^a g(\boldsymbol{\omega}_{\mathbf{k}})$). As this constant tends to be a nuisance, for the remainder of this paper we will consider the asymptotic sampling properties of $\tilde{Q}_{a,\lambda}(g; \mathbf{r})$. We now consider the variance of $\tilde{Q}_{a,\lambda}(g; \mathbf{r})$.

Lemma 3.1 *Suppose Assumptions 2.1, 2.2 and*

(i) Assumption 2.4(i)(a,b) hold. Then we have

$$\lambda^d \text{cov} \left[\tilde{Q}_{a,\lambda}(g; \mathbf{r}_1), \tilde{Q}_{a,\lambda}(g; \mathbf{r}_2) \right] = \begin{cases} C_1(\boldsymbol{\omega}_{\mathbf{r}}) + O(\frac{1}{\lambda} + \frac{\lambda^d}{n}) & \mathbf{r}_1 = \mathbf{r}_2 (= \mathbf{r}) \\ O(\frac{1}{\lambda} + \frac{\lambda^d}{n}) & \mathbf{r}_1 \neq \mathbf{r}_2 \end{cases}$$

and

$$\lambda^d \text{cov} \left[\tilde{Q}_{a,\lambda}(g; \mathbf{r}_1), \overline{\tilde{Q}_{a,\lambda}(g; \mathbf{r}_2)} \right] = \begin{cases} C_2(\boldsymbol{\omega}_{\mathbf{r}}) + O(\frac{1}{\lambda} + \frac{\lambda^d}{n}) & \mathbf{r}_1 = -\mathbf{r}_2 (= \mathbf{r}) \\ O(\frac{1}{\lambda} + \frac{\lambda^d}{n}) & \mathbf{r}_1 \neq -\mathbf{r}_2 \end{cases}$$

where

$$C_1(\boldsymbol{\omega}_{\mathbf{r}}) = \frac{1}{(2\pi)^d} \left(\int_{2\pi[-a/\lambda, a/\lambda]^d} f(\boldsymbol{\omega}) f(\boldsymbol{\omega} + \boldsymbol{\omega}_{\mathbf{r}}) h_1(\boldsymbol{\omega}, \boldsymbol{\omega}_{\mathbf{r}}) d\boldsymbol{\omega} + \int_{\mathcal{D}_{\mathbf{r}}} f(\boldsymbol{\omega}) f(\boldsymbol{\omega} + \boldsymbol{\omega}_{\mathbf{r}}) h_2(\boldsymbol{\omega}, \boldsymbol{\omega}_{\mathbf{r}}) d\boldsymbol{\omega} \right) \quad (12)$$

(noting that $C_1(\boldsymbol{\omega}_{\mathbf{r}})$ is real),

$$C_2(\boldsymbol{\omega}_{\mathbf{r}}) = \frac{1}{(2\pi)^d} \left(\int_{2\pi[-a/\lambda, a/\lambda]^d} f(\boldsymbol{\omega}) f(\boldsymbol{\omega} + \boldsymbol{\omega}_{\mathbf{r}}) h_3(\boldsymbol{\omega}, \boldsymbol{\omega}_{\mathbf{r}}) d\boldsymbol{\omega} + \int_{\mathcal{D}_{\mathbf{r}}} f(\boldsymbol{\omega}) f(\boldsymbol{\omega} + \boldsymbol{\omega}_{\mathbf{r}}) h_4(\boldsymbol{\omega}, \boldsymbol{\omega}_{\mathbf{r}}) d\boldsymbol{\omega} \right), \quad (13)$$

the integral is defined as $\int_{\mathcal{D}_{\mathbf{r}}} = \int_{2\pi \max(-a, -a-r_1)/\lambda}^{2\pi \min(a, a-r_1)/\lambda} \cdots \int_{2\pi \max(-a, -a-r_d)/\lambda}^{2\pi \min(a, a-r_d)/\lambda}$ and

$$h_j(\boldsymbol{\omega}, \boldsymbol{\omega}_{\mathbf{r}}) = \begin{cases} |g(\boldsymbol{\omega})|^2 & j = 1 \\ g(\boldsymbol{\omega}) \overline{g(-\boldsymbol{\omega} - \boldsymbol{\omega}_{\mathbf{r}})} & j = 2 \\ g(\boldsymbol{\omega}) g(-\boldsymbol{\omega}) & j = 3 \\ g(\boldsymbol{\omega}) g(\boldsymbol{\omega} + \boldsymbol{\omega}_{\mathbf{r}}) & j = 4 \end{cases}.$$

(ii) Assumption 2.4(ii)(a) hold, then

$$\lambda^d \text{cov} \left[\tilde{Q}_{a,\lambda}(g; \mathbf{r}_1), \tilde{Q}_{a,\lambda}(g; \mathbf{r}_2) \right] = A_1(\mathbf{r}_1, \mathbf{r}_2) + A_2(\mathbf{r}_1, \mathbf{r}_2) + O\left(\frac{\lambda^d}{n}\right),$$

$$\lambda^d \text{cov} \left[\tilde{Q}_{a,\lambda}(g; \mathbf{r}_1), \overline{\tilde{Q}_{a,\lambda}(g; \mathbf{r}_2)} \right] = A_3(\mathbf{r}_1, \mathbf{r}_2) + A_4(\mathbf{r}_1, \mathbf{r}_2) + O\left(\frac{\lambda^d}{n}\right),$$

where

$$\begin{aligned} A_1(\mathbf{r}_1, \mathbf{r}_2) &= \frac{1}{\pi^{2d} \lambda^d} \sum_{\mathbf{m}=-2a}^{2a} \sum_{\mathbf{k}=\max(-a, -a+\mathbf{m})}^{\min(-a+\mathbf{m})} \int_{\mathbb{R}^{2d}} f\left(\frac{2\mathbf{u}}{\lambda} - \boldsymbol{\omega}_{\mathbf{k}}\right) f\left(\frac{2\mathbf{v}}{\lambda} + \boldsymbol{\omega}_{\mathbf{k}} + \boldsymbol{\omega}_{\mathbf{r}_1}\right) \\ &\quad g(\boldsymbol{\omega}_{\mathbf{k}}) \overline{g(\boldsymbol{\omega}_{\mathbf{k}} - \boldsymbol{\omega}_{\mathbf{m}})} \text{Sinc}(\mathbf{u} + \mathbf{m}\pi) \text{Sinc}(\mathbf{v} - (\mathbf{m} + \mathbf{r}_1 - \mathbf{r}_2)\pi) \text{Sinc}(\mathbf{u}) \text{Sinc}(\mathbf{v}) d\mathbf{u} d\mathbf{v} \\ A_2(\mathbf{r}_1, \mathbf{r}_2) &= \frac{1}{\pi^{2d} \lambda^d} \sum_{\mathbf{m}=-2a}^{2a} \sum_{\mathbf{k}=\max(-a, -a-\mathbf{m})}^{\min(-a-\mathbf{m})} \int_{\mathbb{R}^{2d}} f\left(\frac{2\mathbf{u}}{\lambda} - \boldsymbol{\omega}_{\mathbf{k}}\right) f\left(\frac{2\mathbf{v}}{\lambda} + \boldsymbol{\omega}_{\mathbf{k}} + \boldsymbol{\omega}_{\mathbf{r}_1}\right) \\ &\quad g(\boldsymbol{\omega}_{\mathbf{k}}) \overline{g(\boldsymbol{\omega}_{\mathbf{m}} - \boldsymbol{\omega}_{\mathbf{k}})} \text{Sinc}(\mathbf{u} + (\mathbf{m} + \mathbf{r}_1)\pi) \text{Sinc}(\mathbf{v} + (\mathbf{m} + \mathbf{r}_2)\pi) \text{Sinc}(\mathbf{u}) \text{Sinc}(\mathbf{v}) d\mathbf{u} d\mathbf{v} \\ A_3(\mathbf{r}_1, \mathbf{r}_2) &= \frac{1}{\pi^{2d} \lambda^d} \sum_{\mathbf{m}=-2a}^{2a} \sum_{\mathbf{k}=\max(-a, -a+\mathbf{m})}^{\min(-a+\mathbf{m})} \int_{\mathbb{R}^{2d}} f\left(\frac{2\mathbf{u}}{\lambda} - \boldsymbol{\omega}_{\mathbf{k}}\right) f\left(\frac{2\mathbf{v}}{\lambda} + \boldsymbol{\omega}_{\mathbf{k}} + \boldsymbol{\omega}_{\mathbf{r}_1}\right) \\ &\quad g(\boldsymbol{\omega}_{\mathbf{k}}) \overline{g(\boldsymbol{\omega}_{\mathbf{m}} - \boldsymbol{\omega}_{\mathbf{k}})} \text{Sinc}(\mathbf{u} + \mathbf{m}\pi) \text{Sinc}(\mathbf{v} + (\mathbf{m} + \mathbf{r}_2 + \mathbf{r}_1)\pi) \text{Sinc}(\mathbf{u}) \text{Sinc}(\mathbf{v}) d\mathbf{u} d\mathbf{v} \\ A_4(\mathbf{r}_1, \mathbf{r}_2) &= \frac{1}{\pi^{2d} \lambda^d} \sum_{\mathbf{m}=-2a}^{2a} \sum_{\mathbf{k}=\max(-a, -a-\mathbf{m})}^{\min(-a-\mathbf{m})} \int_{\mathbb{R}^{2d}} f\left(\frac{2\mathbf{u}}{\lambda} - \boldsymbol{\omega}_{\mathbf{k}}\right) f\left(\frac{2\mathbf{v}}{\lambda} + \boldsymbol{\omega}_{\mathbf{k}} + \boldsymbol{\omega}_{\mathbf{r}_1}\right) \\ &\quad g(\boldsymbol{\omega}_{\mathbf{k}}) \overline{g(\boldsymbol{\omega}_{\mathbf{k}} - \boldsymbol{\omega}_{\mathbf{m}})} \text{Sinc}(\mathbf{u} - (\mathbf{m} + \mathbf{r}_2)\pi) \text{Sinc}(\mathbf{v} + (\mathbf{m} + \mathbf{r}_1)\pi) \text{Sinc}(\mathbf{u}) \text{Sinc}(\mathbf{v}) d\mathbf{u} d\mathbf{v} \end{aligned}$$

where $\mathbf{k} = \max(-a, -a-\mathbf{m}) = \{k_1 = \max(-a, -a-m_1), \dots, k_d = \max(-a, -a-m_d)\}$,
 $\mathbf{k} = \min(-a + \mathbf{m}) = \{k_1 = \max(-a, -a + m_1), \dots, k_d = \min(-a + m_d)\}$.

(iii) Assumption 2.4(i) or 2.4(ii)(a) hold. Then we have

$$\lambda^d \sup_a \left| \text{cov} \left[\tilde{Q}_{a,\lambda}(g; \mathbf{r}_1), \tilde{Q}_{a,\lambda}(g; \mathbf{r}_2) \right] \right| < \infty \quad \lambda^d \sup_a \left| \text{cov} \left[\tilde{Q}_{a,\lambda}(g; \mathbf{r}_1), \overline{\tilde{Q}_{a,\lambda}(g; \mathbf{r}_2)} \right] \right| < \infty,$$

if $\lambda^d/n \rightarrow 0$ as $\lambda \rightarrow \infty$ and $n \rightarrow \infty$.

PROOF See Section 6. □

From Lemma 3.1 we observe though the variance for both the fixed and increasing frequency domain have the same order (see Lemma 3.1(iii)), their expressions are very different. For example, if the domain is kept fixed, the sequence of random variables $\{\tilde{Q}_{a,\lambda}(g; \mathbf{r}); \mathbf{r} \in \mathbb{Z}^d\}$ are asymptotically uncorrelated, in contrast this property does not in increasing frequency domain set-up. In the following theorem we reconcile these differences by placing some conditions on the spectral density estimator (see Assumption 2.4(b)) and imposing some mild restrictions on the rate of grow of the frequency domain a .

Theorem 3.2 [Asymptotic expression for variance] Suppose Assumptions 2.1, 2.2 and 2.4(ii)(a,b) hold. Then we have

$$\lambda^d \text{cov} \left[\tilde{Q}_{a,\lambda}(g; \mathbf{r}_1), \tilde{Q}_{a,\lambda}(g; \mathbf{r}_2) \right] = \begin{cases} C_1(\boldsymbol{\omega}_{\mathbf{r}}) + O(\ell_{\lambda,a,n}) & \mathbf{r}_1 = \mathbf{r}_2 (= \mathbf{r}) \\ O(\ell_{\lambda,a,n}) & \mathbf{r}_1 \neq \mathbf{r}_2 \end{cases}$$

$$\lambda^d \text{cov} \left[\tilde{Q}_{a,\lambda}(g; \mathbf{r}_1), \overline{\tilde{Q}_{a,\lambda}(g; \mathbf{r}_2)} \right] = \begin{cases} C_2(\boldsymbol{\omega}_{\mathbf{r}}) + O(\ell_{\lambda,a,n}) & \mathbf{r}_1 = -\mathbf{r}_2 (= \mathbf{r}) \\ O(\ell_{\lambda,a,n}) & \mathbf{r}_1 \neq -\mathbf{r}_2 \end{cases},$$

and

$$\ell_{\lambda,a,n} = \log^2(a) \Gamma(a/\lambda) \left[\frac{(\log a + \log \lambda)}{\lambda} \right] + \frac{\lambda^d}{n}.$$

PROOF See Section 6. □

Lemma 3.2 Suppose Assumption 2.4(ii)(a) holds, and \mathbf{r} is fixed. Then we have

$$C_j(\boldsymbol{\omega}_{\mathbf{r}}) = C_j + O\left(\frac{\|\mathbf{r}\|_1}{\lambda}\right),$$

where

$$C_1 = \frac{1}{(2\pi)^d} \int_{2\pi[-a/\lambda, a/\lambda]^d} f(\boldsymbol{\omega})^2 \left(|g(\boldsymbol{\omega})|^2 + g(\boldsymbol{\omega}) \overline{g(-\boldsymbol{\omega})} \right) d\boldsymbol{\omega}$$

$$C_2 = \frac{1}{(2\pi)^d} \int_{2\pi[-a/\lambda, a/\lambda]^d} f(\boldsymbol{\omega})^2 (g(\boldsymbol{\omega})g(-\boldsymbol{\omega}) + g(\boldsymbol{\omega})g(\boldsymbol{\omega})) d\boldsymbol{\omega}$$

PROOF. See Section 6. □

The asymptotic expression for the variance was obtained under the assumption that the spatial process is Gaussian. In the following theorem, we extend these results to non-Gaussian random fields. We make the assumption that $\{Z(\mathbf{s}); \mathbf{s} \in \mathbb{R}^d\}$ is fourth order stationary, in the sense that $\text{cov}[Z(\mathbf{s}_1), Z(\mathbf{s}_2)] = c(\mathbf{s}_1 - \mathbf{s}_2)$ and $\text{cum}[Z(\mathbf{s}_1), Z(\mathbf{s}_2), Z(\mathbf{s}_3), Z(\mathbf{s}_4)] = \kappa_4(\mathbf{s}_1 - \mathbf{s}_2, \mathbf{s}_1 - \mathbf{s}_3, \mathbf{s}_1 - \mathbf{s}_4)$, for some function $\kappa_4(\cdot)$ and all $\mathbf{s}_1, \dots, \mathbf{s}_4 \in \mathbb{R}^d$.

Theorem 3.3 Let us suppose that $\{Z(\mathbf{s}); \mathbf{s} \in \mathbb{R}^d\}$ is a fourth order stationary spatial random field that satisfies Assumption 2.1(i). Furthermore, we suppose that Assumptions 2.2 and 2.4(ii)(a,b) are satisfied. In addition, let $\kappa_4(\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3) = \text{cum}(Z(0), Z(\mathbf{s}_1), Z(\mathbf{s}_2), Z(\mathbf{s}_3))$ and $f_4(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3) = \int_{\mathbb{R}^{3d}} \kappa_4(\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3) \exp(-i \sum_{j=1}^3 \mathbf{s}'_j \boldsymbol{\omega}_j) d\boldsymbol{\omega}_1 d\boldsymbol{\omega}_2 d\boldsymbol{\omega}_3$. We assume that the spatial tri-spectral density function is such that $\int_{\mathbb{R}^{3d}} |f_4(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3)| d\boldsymbol{\omega}_1 d\boldsymbol{\omega}_2 d\boldsymbol{\omega}_3 < \infty$ and for

$1 \leq j \leq 3d$ there exists a $\beta_j : \mathbb{R}^{3d} \rightarrow \mathbb{R}$ such that $|\frac{\partial f_4(\omega_1, \dots, \omega_{3d})}{\partial \omega_i}| \leq \beta_j(\omega_1, \dots, \omega_j, \dots, \omega_{3d})$ where $\beta_j(\cdot)$ is monotonically decreasing, as defined in Assumption 2.4(ii)(b). Furthermore, $\int_{\mathbb{R}^{d-1}} \int_0^b \beta_j(\omega_1, \dots, \omega_j, \dots, \omega_{3d}) d\omega_j d\omega_1 \dots d\omega_{3d} \leq \Gamma(b)$, where $\Gamma(\cdot)$ is defined in Assumption 2.4(ii)(b).

If $a^d = O(n)$ and \mathbf{r} is fixed, then we have

$$\lambda^d \text{cov} [\tilde{Q}_{a,\lambda}(g; \mathbf{r}_1), \tilde{Q}_{a,\lambda}(g; \mathbf{r}_2)] = \begin{cases} C_1 + D_1 + O(\ell_{\lambda,a,n}^{(2)} + \frac{\|\mathbf{r}\|_1}{\lambda}) & \mathbf{r}_1 = \mathbf{r}_2 (= \mathbf{r}) \\ O(\ell_{\lambda,a,n}^{(2)}) & \mathbf{r}_1 \neq \mathbf{r}_2 \end{cases} \quad (14)$$

$$\lambda^d \text{cov} [\tilde{Q}_{a,\lambda}(g; \mathbf{r}_1), \overline{\tilde{Q}_{a,\lambda}(g; \mathbf{r}_2)}] = \begin{cases} C_2 + D_2 + O(\ell_{\lambda,a,n}^{(2)} + \frac{\|\mathbf{r}\|_1}{\lambda}) & \mathbf{r}_1 = -\mathbf{r}_2 (= \mathbf{r}) \\ O(\ell_{\lambda,a,n}^{(2)}) & \mathbf{r}_1 \neq -\mathbf{r}_2 \end{cases}, \quad (15)$$

where C_1 and C_2 are defined in Lemma 3.2,

$$\begin{aligned} D_1 &= \frac{1}{(2\pi)^d} \int_{2\pi[-a/\lambda, a/\lambda]^{2d}} g(\boldsymbol{\omega}_1) \overline{g(\boldsymbol{\omega}_2)} f_4(-\boldsymbol{\omega}_1, -\boldsymbol{\omega}_2, \boldsymbol{\omega}_2) d\boldsymbol{\omega}_1 d\boldsymbol{\omega}_2 \\ D_2 &= \frac{1}{(2\pi)^d} \int_{2\pi[-a/\lambda, a/\lambda]^2} g(\boldsymbol{\omega}_1) g(\boldsymbol{\omega}_2) f_4(-\boldsymbol{\omega}_1, -\boldsymbol{\omega}_2, \boldsymbol{\omega}_2) d\boldsymbol{\omega}_1 d\boldsymbol{\omega}_2 \end{aligned}$$

and $\ell_{\lambda,a,n}^{(2)} = \ell_{\lambda,a,n} + \frac{a\lambda^d}{n^2} + \Gamma(a/\lambda) \frac{\log^3(a)}{\lambda}$.

In the following theorem we derive bounds for the cumulants of $\tilde{Q}_{a,\lambda}(g; \mathbf{r})$, which is subsequently used to show asymptotical normality of $\tilde{Q}_{a,\lambda}(g; \mathbf{r})$.

Theorem 3.4 Suppose Assumptions 2.1, 2.2, and 2.4(i)(a,b) or (ii)(a) hold. Then we have

$$\text{cum}_q(\tilde{Q}_{a,\lambda}(g, \mathbf{r}_1), \dots, \tilde{Q}_{a,\lambda}(g, \mathbf{r}_q)) = O\left(\frac{1}{\log^2(a)} \left[\frac{\log^2(a)}{\lambda}\right]^{d(q-1)}\right) \quad (16)$$

if $\frac{\lambda^d}{n \log^{2d}(a)} \rightarrow 0$ as $n \rightarrow \infty$, $a \rightarrow \infty$ and $\lambda \rightarrow \infty$.

PROOF See Section A.3. □

From the above theorem we see that if $\frac{\lambda^d}{n \log^{2d}(a)} \rightarrow 0$ and $\log^2(a)/\lambda^{1/2} \rightarrow 0$ as $\lambda \rightarrow \infty$, $n \rightarrow \infty$ and $a \rightarrow \infty$, then we have $\lambda^{dq/2} \text{cum}_q(\tilde{Q}_{a,\lambda}(g, \mathbf{r}_1), \dots, \tilde{Q}_{a,\lambda}(g, \mathbf{r}_q)) \rightarrow 0$ for all orders of cumulants q . Using this result we obtain asymptotically Gaussian of $\tilde{Q}_{a,\lambda}(g, \mathbf{r})$.

Theorem 3.5 [CLT on real and imaginary parts] Suppose Assumptions 2.1, 2.2 and 2.4(i)(a,b) or 2.4(ii)(a,b) hold. Let C_1 be defined in Lemma 3.2. Then, if $\mathbf{r}_1, \dots, \mathbf{r}_m$ are such that

$\mathbf{r}_i \neq -\mathbf{r}_j$ for any $i, j \in \{1, \dots, m\}$ we have

$$\frac{\lambda^{d/2}}{C_1} \begin{pmatrix} \Re \tilde{Q}_{a,\lambda}(g, \mathbf{r}_1) \\ \vdots \\ \Re \tilde{Q}_{a,\lambda}(g, \mathbf{r}_m) \\ \Im \tilde{Q}_{a,\lambda}(g, \mathbf{r}_1) \\ \vdots \\ \Im \tilde{Q}_{a,\lambda}(g, \mathbf{r}_m) \end{pmatrix} \xrightarrow{\mathcal{P}} \mathcal{N}(0, I_{2d}),$$

with $\frac{\lambda^d}{n \log^{2d}(a)} \rightarrow 0$, $\frac{\log^3(a)}{\lambda^{1/2}} \rightarrow 0$ and $\Gamma(a/\lambda) \frac{\log a}{\lambda} \rightarrow 0$ as $\lambda \rightarrow \infty$, $n \rightarrow \infty$ and $a \rightarrow \infty$.

PROOF See Sections 6 and A.3. □

It is highly likely that the above result also holds when the assumption of Gaussianity of the spatial random field is relaxed and replaced with the assumption the conditions stated in Theorem 3.3 together with some mixing-type assumptions. However, this is beyond the current paper.

4 Sampling properties of $\tilde{Q}_{a,\lambda}(g; \mathbf{r})$ for non-uniform sampling

In this section we consider the asymptotic sampling properties of $\tilde{Q}_{a,\lambda}(g; \mathbf{r})$ when the locations $\{\mathbf{s}_j\}_j$ are not uniformly sampled on $[-\lambda/2, \lambda/2]^d$. We will assume that $\{\mathbf{s}_j\}$ are iid random variables which satisfy Assumption 2.3.

The proof of the results below are based on the methodology developed in the case of uniform sampling. We give a flavour of the proof and its connection to the uniform case by considering the covariance between the DFTs. It is straightforward to show that

$$\begin{aligned} & \text{cov} [J_T(\boldsymbol{\omega}_{\mathbf{k}_1}), J_T(\boldsymbol{\omega}_{\mathbf{k}_2})] \\ &= \frac{C_2}{\lambda^d} \int_{[-\lambda/2, \lambda/2]^d} \int_{[-\lambda/2, \lambda/2]^d} c(\mathbf{s}_1 - \mathbf{s}_2) h\left(\frac{\mathbf{s}_1}{\lambda}\right) h\left(\frac{\mathbf{s}_2}{\lambda}\right) e^{i\mathbf{s}'_1 \boldsymbol{\omega}_{\mathbf{k}_1} - i\mathbf{s}'_2 \boldsymbol{\omega}_{\mathbf{k}_2}} d\mathbf{s}_1 d\mathbf{s}_2 \\ &+ \frac{c(0)\lambda^d}{n} \int_{[-1/2, 1/2]^d} h(\mathbf{u}) e^{-i2\mathbf{u}'\pi(\mathbf{k}_1 - \mathbf{k}_2)} d\mathbf{u}. \end{aligned} \tag{17}$$

Replacing the $h(\cdot)$ s with their Fourier representations gives

$$\begin{aligned} & \text{cov}[J_T(\boldsymbol{\omega}_{\mathbf{k}_1}), J_T(\boldsymbol{\omega}_{\mathbf{k}_2})] = \\ &= C_2 \sum_{j_1, j_2 = -\infty}^{\infty} \gamma_{j_1} \gamma_{j_2} \frac{1}{\lambda^d} \int_{[-\lambda/2, \lambda/2]^d} \int_{[-\lambda/2, \lambda/2]^d} c(\mathbf{s}_1 - \mathbf{s}_2) e^{i\mathbf{s}_1(\boldsymbol{\omega}_{j_1} + \boldsymbol{\omega}_{\mathbf{k}_1}) - i\mathbf{s}_2(\boldsymbol{\omega}_{\mathbf{k}_2} - \boldsymbol{\omega}_{j_2})} d\mathbf{s}_1 d\mathbf{s}_2 \\ & \quad + \frac{c(0)\gamma_{\mathbf{k}_1 - \mathbf{k}_2} \lambda^d}{n}. \end{aligned}$$

We observe that the integrals within $\text{cov}[J_T(\boldsymbol{\omega}_{\mathbf{k}_1}), J_T(\boldsymbol{\omega}_{\mathbf{k}_2})]$ are the same as in the uniform case, just with the additional exponential term. This demonstrates, that the techniques developed for uniform sampling also apply to non-uniform sampling.

Lemma 4.1 *Suppose that $\{Z(\mathbf{s}); \mathbf{s} \in \mathbb{R}^d\}$ is a spatially stationary random fields and Assumptions 2.1 and 2.3 hold.*

(i) *If in addition Assumption 2.4(i)(a,b) holds, then we have*

$$\mathbb{E}[\tilde{Q}_{\lambda,a}(g; \mathbf{r})] = \langle \gamma, \gamma_{-\mathbf{r}} \rangle \frac{1}{(2\pi)^d} \int_{2\pi[-C, C]^d} g(\boldsymbol{\omega}) f(\boldsymbol{\omega}) d\boldsymbol{\omega} + O\left(\frac{1}{\lambda}\right)$$

$$\text{where } \langle \gamma, \gamma_{-\mathbf{r}} \rangle = \sum_{\mathbf{j} \in \mathbb{Z}^d} \gamma_{\mathbf{j}} \gamma_{\mathbf{r} - \mathbf{j}}.$$

(ii) *On the other hand, if Assumption 2.4(ii)(a,b) holds, then we have*

$$\mathbb{E}[\tilde{Q}_{\lambda,a}(g; \mathbf{r})] = \langle \gamma, \gamma_{-\mathbf{r}} \rangle \frac{1}{(2\pi)^d} \int_{2\pi[-a/\lambda, a/\lambda]^d} g(\boldsymbol{\omega}) f(\boldsymbol{\omega}) d\boldsymbol{\omega} + O\left(\frac{\Gamma(a/\lambda)(\log \lambda + \log a)}{\lambda}\right)$$

PROOF See Section 7. □

We observe that despite the DFTs not being near uncorrelated, Lemma 4.1 shows that

$$\mathbb{E}[\tilde{Q}_{\lambda,a}(g; 0)] \approx \frac{1}{(2\pi)^d} \int_{2\pi[-a/\lambda, a/\lambda]^d} g(\boldsymbol{\omega}) f(\boldsymbol{\omega}) d\boldsymbol{\omega}$$

for $\lambda \rightarrow \infty$ (since $\sum_{\mathbf{j} \in \mathbb{Z}^d} |\gamma_{\mathbf{j}}|^2 = 1$). This means that $\tilde{Q}_{\lambda,a}(g; 0)$ is asymptotically an unbiased estimator of $\int_{[-a/\lambda, a/\lambda]^d} g(\boldsymbol{\omega}) f(\boldsymbol{\omega}) d\boldsymbol{\omega}$, regardless of the sampling scheme of the locations.

Theorem 4.1 *Suppose Assumptions 2.1, 2.3,*

(i) and Assumption 2.4(i)(a,b) hold. Then we have

$$\begin{aligned} \lambda^d \text{cov}(\tilde{Q}_{a,\lambda}(g; \mathbf{r}_1), \tilde{Q}_{a,\lambda}(g; \mathbf{r}_2)) &= U_{\lambda}^{(1)}(\mathbf{r}_1, \mathbf{r}_2) + U_{\lambda}^{(2)}(\mathbf{r}_1, \mathbf{r}_2) + O\left(\frac{1 + \|\mathbf{r}_1\|_1 + \|\mathbf{r}_2\|_1}{\lambda} + \frac{\lambda^d}{n}\right) \\ \lambda^d \text{cov}(\tilde{Q}_{a,\lambda}(g; \mathbf{r}_1), \overline{\tilde{Q}_{a,\lambda}(g; \mathbf{r}_2)}) &= U_{\lambda}^{(3)}(\mathbf{r}_1, \mathbf{r}_2) + U_{\lambda}^{(4)}(\mathbf{r}_1, \mathbf{r}_2) + O\left(\frac{1 + \|\mathbf{r}_1\|_1 + \|\mathbf{r}_2\|_1}{\lambda} + \frac{\lambda^d}{n}\right) \end{aligned}$$

(ii) and Assumption 2.4(ii)(a,b) hold. Then we have

$$\begin{aligned}\lambda^d \text{cov}(\tilde{Q}_{a,\lambda}(g; \mathbf{r}_1), \tilde{Q}_{a,\lambda}(g; \mathbf{r}_2)) &= U_\lambda^{(1)}(\mathbf{r}_1, \mathbf{r}_2) + U_\lambda^{(2)}(\mathbf{r}_1, \mathbf{r}_2) + O(\ell_{\lambda,a,n} + \frac{1 + \|\mathbf{r}_1\|_1 + \|\mathbf{r}_2\|_1}{\lambda}) \\ \lambda^d \text{cov}(\tilde{Q}_{a,\lambda}(g; \mathbf{r}_1), \overline{\tilde{Q}_{a,\lambda}(g; \mathbf{r}_2)}) &= U_\lambda^{(3)}(\mathbf{r}_1, \mathbf{r}_2) + U_\lambda^{(4)}(\mathbf{r}_1, \mathbf{r}_2) + O(\ell_{\lambda,a,n} + \frac{1 + \|\mathbf{r}_1\|_1 + \|\mathbf{r}_2\|_1}{\lambda})\end{aligned}$$

where

$$\begin{aligned}U_\lambda^{(1)}(\mathbf{r}_1, \mathbf{r}_2) &= \frac{1}{(2\pi)^d} \sum_{\mathbf{j}_1 + \dots + \mathbf{j}_4 = \mathbf{r}_2 - \mathbf{r}_1} \gamma_{\mathbf{j}_1} \gamma_{\mathbf{j}_2} \gamma_{\mathbf{j}_3} \gamma_{\mathbf{j}_4} \int_{2\pi[-a/\lambda, a/\lambda]^d} |g(\boldsymbol{\omega})|^2 f(\boldsymbol{\omega})^2 d\boldsymbol{\omega} \\ U_\lambda^{(2)}(\mathbf{r}_1, \mathbf{r}_2) &= \frac{1}{(2\pi)^d} \sum_{\mathbf{j}_1 + \dots + \mathbf{j}_4 = \mathbf{r}_2 - \mathbf{r}_1} \gamma_{\mathbf{j}_1} \gamma_{\mathbf{j}_2} \gamma_{\mathbf{j}_3} \gamma_{\mathbf{j}_4} \int_{2\pi[-a/\lambda, a/\lambda]^d} g(\boldsymbol{\omega}) \overline{g(-\boldsymbol{\omega})} f(\boldsymbol{\omega})^2 d\boldsymbol{\omega} \\ U_\lambda^{(3)}(\mathbf{r}_1, \mathbf{r}_2) &= \frac{1}{(2\pi)^d} \sum_{\mathbf{j}_1 + \dots + \mathbf{j}_4 = \mathbf{r}_2 + \mathbf{r}_1} \gamma_{\mathbf{j}_1} \gamma_{\mathbf{j}_2} \gamma_{\mathbf{j}_3} \gamma_{\mathbf{j}_4} \int_{2\pi[-a/\lambda, a/\lambda]^d} g(\boldsymbol{\omega})^2 f(\boldsymbol{\omega})^2 d\boldsymbol{\omega} \\ U_\lambda^{(4)}(\mathbf{r}_1, \mathbf{r}_2) &= \frac{1}{(2\pi)^d} \sum_{\mathbf{j}_1 + \dots + \mathbf{j}_4 = \mathbf{r}_2 + \mathbf{r}_1} \gamma_{\mathbf{j}_1} \gamma_{\mathbf{j}_2} \gamma_{\mathbf{j}_3} \gamma_{\mathbf{j}_4} \int_{2\pi[-a/\lambda, a/\lambda]^d} g(\boldsymbol{\omega}) g(-\boldsymbol{\omega}) f(\boldsymbol{\omega})^2 d\boldsymbol{\omega}\end{aligned}$$

PROOF See Section 7. □

Applying Theorem 4.1(i) to the Whittle likelihood, (3), we observe that the asymptotic sampling properties of (3) are identical to integral Whittle likelihood considered in Matsuda and Yajima (2009).

5 Proof of Theorem 3.1

It is clear from the motivation at the start of Section 3 that the sinc function plays an important role in the analysis of $Q_{a,\lambda}(g; \mathbf{r})$ and $\tilde{Q}_{a,\lambda}(g; \mathbf{r})$. Therefore we now summarise some of its properties. It is well known that

$$\int_{-\infty}^{\infty} \text{sinc}(u) du = \pi \text{ and } \int_{-\infty}^{\infty} \text{sinc}^2(u) du = \pi. \quad (18)$$

We next state a well known result that is an important component of the proofs in this paper.

Lemma 5.1 [*Orthogonality of the sinc function*]

$$\int_{-\infty}^{\infty} \text{sinc}(u) \text{sinc}(u + x) du = \pi \text{sinc}(x) \quad (19)$$

and if $s \in \mathbb{Z}/\{0\}$ then

$$\int_{-\infty}^{\infty} \text{sinc}(u) \text{sinc}(u + s\pi) du = 0. \quad (20)$$

PROOF In Appendix A.1. □

We use the above result in the proof of Lemma 3.1.

PROOF of Lemma 3.1. We first prove (i). Since $a = O(\lambda)$, by taking expectations and using Theorem 2.1 we have

$$\mathbb{E}[Q_{a,\lambda}(g; \mathbf{r})] = \begin{cases} \frac{1}{\lambda^d} \sum_{\mathbf{k}=-a}^a f(\boldsymbol{\omega}_{\mathbf{k}}) + O(\frac{1}{\lambda} + \frac{\lambda^d}{n}) & \mathbf{r} = 0 \\ O(\frac{1}{\lambda^{d-b}}) & \mathbf{r} \neq 0 \text{ but } b \text{ elements of the} \\ & d\text{-dimensional vector } \mathbf{r} \text{ are equal to zero} \end{cases}$$

Therefore, by replacing the summand with the integral we obtain (9). The proof of (8) uses the proof of Theorem 2.1 in Bandyopadhyay and Subba Rao (2014) and is very similar to the above.

The above method cannot be used to prove (ii) since $a/\lambda \rightarrow \infty$, this leads to bounds which may not converge. Therefore, as discussed in Section 3 we consider an alternative approach. We first prove (11). We expand $Q_{a,\lambda}(g; \mathbf{r})$ as a quadratic form to give

$$Q_{a,\lambda}(g; \mathbf{r}) = \frac{1}{n^2} \sum_{j_1, j_2=1}^n \sum_{\mathbf{k}=-a}^a g(\boldsymbol{\omega}_{\mathbf{k}}) Z(\mathbf{s}_{j_1}) Z(\mathbf{s}_{j_2}) \exp(i\boldsymbol{\omega}'_{\mathbf{k}}(\mathbf{s}_{j_1} - \mathbf{s}_{j_2})) \exp(-i\boldsymbol{\omega}'_{\mathbf{r}} \mathbf{s}_{j_2}).$$

Taking expectations gives

$$\begin{aligned} \mathbb{E}[Q_{a,\lambda}(g; \mathbf{r})] &= C_2 \sum_{\mathbf{k}=-a}^a g(\boldsymbol{\omega}_{\mathbf{k}}) \mathbb{E}[c(\mathbf{s}_1 - \mathbf{s}_2) \exp(i\boldsymbol{\omega}'_{\mathbf{k}}(\mathbf{s}_1 - \mathbf{s}_2) - i\mathbf{s}'_2 \boldsymbol{\omega}_{\mathbf{r}})] \\ &\quad + \frac{c(0)}{n} \sum_{\mathbf{k}=-a}^a g(\boldsymbol{\omega}_{\mathbf{k}}) \mathbb{E}[\exp(-i\boldsymbol{\omega}'_{\mathbf{r}} \mathbf{s})]. \end{aligned} \quad (21)$$

In the case that $d = 1$ the above becomes

$$\begin{aligned} \mathbb{E}[Q_{a,\lambda}(g; r)] &= C_2 \sum_{k=-a}^a g(\omega_k) \mathbb{E}[c(s_1 - s_2) \exp(i\omega_k(s_1 - s_2) - is_2 \omega_r)] + \frac{c(0)}{n} \sum_{k=-a}^a g(\omega_k) I(r=0) \\ &= C_2 \sum_{k=-a}^a g(\omega_k) \mathbb{E}[c(s_1 - s_2) \exp(i\omega_k(s_1 - s_2) - is_2 \omega_r)] + V_r \\ &= \frac{C_2}{\lambda^2} \sum_{k=-a}^a g(\omega_k) \int_{-\lambda/2}^{\lambda/2} \int_{-\lambda/2}^{\lambda/2} c(s_1 - s_2) \exp(i\omega_k(s_1 - s_2) - is_2 \omega_r) ds_1 ds_2 + V_r, \end{aligned} \quad (22)$$

where $V_r = \frac{c(0)}{n} \sum_{k=-a}^a g(\omega_k) I(r=0)$. Replacing $c(s_1 - s_2)$ with the Fourier representation of the covariance function gives

$$\mathbb{E}[Q_{a,\lambda}(g; r)] = \frac{C_2}{2\pi} \sum_{k=-a}^a g(\omega_k) \int_{-\infty}^{\infty} f(\omega) \text{sinc}(\frac{\lambda\omega}{2} + k\pi) \text{sinc}(\frac{\lambda\omega}{2} + (k+r)\pi) d\omega + V_r.$$

By a change of variables $y = \lambda\omega/2 + k\pi$ and replacing the sum with an integral we have

$$\begin{aligned} \mathbb{E}[Q_{a,\lambda}(g; r)] &= \frac{C_2}{\pi\lambda} \sum_{k=-a}^a g(\omega_k) \int_{-\infty}^{\infty} f\left(\frac{2y}{\lambda} - \omega_k\right) \text{sinc}(y) \text{sinc}(y + r\pi) dy + I(r=0) \frac{1}{n} \sum_{k=-a}^a g(\omega_k) \\ &= \frac{C_2}{\pi} \int_{-\infty}^{\infty} \text{sinc}(y) \text{sinc}(y + r\pi) \left(\frac{1}{\lambda} \sum_{k=-a}^a g(\omega_k) f\left(\frac{2y}{\lambda} - \omega_k\right) \right) dy + V_r \\ &= \frac{C_2}{2\pi} \int_{-\infty}^{\infty} \text{sinc}(y) \text{sinc}(y + r\pi) \int_{-2\pi a/\lambda}^{2\pi a/\lambda} g(u) f\left(\frac{2y}{\lambda} - u\right) du dy + V_r + O\left(\frac{1}{\lambda}\right). \end{aligned}$$

Replacing $f(\frac{2y}{\lambda} - u)$ with $f(-u)$ and C_2 with one gives

$$\begin{aligned} &\mathbb{E}[Q_{a,\lambda}(g; r)] \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{sinc}(y) \text{sinc}(y + r\pi) \int_{-2\pi a/\lambda}^{2\pi a/\lambda} g(u) f(-u) du dy + R_n + V_r + O\left(\frac{1}{\lambda} + \frac{1}{n}\right) \end{aligned}$$

where

$$R_n = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{sinc}(y) \text{sinc}(y + r\pi) \left(\int_{-2\pi a/\lambda}^{2\pi a/\lambda} g(u) \left(f\left(\frac{2y}{\lambda} - u\right) - f(-u) \right) du \right) dy.$$

By using Lemma A.2 we have $|R_n| = O(\Gamma(a/\lambda) \frac{(\log \lambda + \log a)}{\lambda})$ (since $a \gg \lambda$). Therefore, by using Lemma 5.1 and (18) we have

$$\begin{aligned} &\mathbb{E}[Q_{a,\lambda}(g; r)] \\ &= \frac{1}{2\pi^2} \int_{-\infty}^{\infty} \text{sinc}(y) \text{sinc}(y + r\pi) \int_{-2\pi a/\lambda}^{2\pi a/\lambda} g(u) f(-2\pi u) du dy + V_r + O\left(\frac{1}{\lambda} + \frac{\lambda}{a} + \frac{\log(\lambda + |a|)}{\lambda} + \frac{1}{n}\right) \\ &= I(r=0) \left(\frac{1}{2\pi} \int_{-2\pi a/\lambda}^{2\pi a/\lambda} g(u) f(u) du + \frac{c(0)}{n} \sum_{k=-a}^a g(\omega_k) \right) + O\left(\frac{1}{\lambda} + \frac{\Gamma(a/\lambda)(\log \lambda + \log a)}{\lambda} + \frac{1}{n}\right), \end{aligned}$$

which gives (11) in the case $d = 1$.

To prove the result for $d > 1$, we note that by substituting the spectral representation into (21) and replacing sum with integral we have

$$\begin{aligned} &\mathbb{E}[Q_{a,\lambda}(g; \mathbf{r})] \\ &= \frac{C_2}{\pi^d \lambda^d} \sum_{\mathbf{k}=-a}^a g(\omega_{\mathbf{k}}) \int_{\mathbb{R}^d} f\left(\frac{2\mathbf{u}}{\lambda} - \omega_{\mathbf{k}}\right) \text{Sinc}(\mathbf{u}) \text{Sinc}(\mathbf{u} + \mathbf{r}\pi) d\mathbf{u} + V_{\mathbf{r}} \\ &= \frac{C_2}{(2\pi)^d \pi^d} \int_{\mathbb{R}^d} \text{Sinc}(\mathbf{u}) \text{Sinc}(\mathbf{u} + \mathbf{r}\pi) \int_{2\pi[-a/\lambda, a/\lambda]^d} g(\omega) f\left(\frac{2\mathbf{u}}{\lambda} - \omega\right) d\omega d\mathbf{u} + V_{\mathbf{r}} + O\left(\frac{1}{\lambda^d}\right), \end{aligned}$$

where $V_{\mathbf{r}} = I(\mathbf{r}=0) \frac{c(0)}{n} \sum_{\mathbf{k}=-a}^a g(\omega_{\mathbf{k}})$. Now we follow the identical method to the univariate case and replace $f(\frac{2u_1}{\lambda} - \omega_1, \dots, \frac{2u_d}{\lambda} - \omega_d)$ with $f(-\omega_1, \dots, -\omega_d)$, this gives

$$\begin{aligned} &\mathbb{E}(Q_{a,\lambda}(g; \mathbf{r})) \\ &= \frac{1}{(2\pi)^d \pi^d} \int_{\mathbb{R}^d} \text{Sinc}(\mathbf{u}) \text{Sinc}(\mathbf{u} + \mathbf{r}\pi) \int_{2\pi[-a/\lambda, a/\lambda]^d} g(\omega) f(-\omega) d\omega d\mathbf{u} + R_n + V_{\mathbf{r}} + O\left(\frac{1}{\lambda^d} + \frac{1}{n}\right), \end{aligned}$$

where

$$\begin{aligned}
R_n &= \frac{1}{(2\pi)^d \pi^d} \int_{\mathbb{R}^d} \text{Sinc}(\mathbf{u}) \text{Sinc}(\mathbf{u} + \mathbf{r}\pi) \int_{2\pi[-a/\lambda, a/\lambda]^d} g(\boldsymbol{\omega}) \left(f\left(\frac{2\mathbf{u}}{\lambda} - \boldsymbol{\omega}\right) - f(\boldsymbol{\omega}) \right) d\boldsymbol{\omega} d\mathbf{u} \\
&= \frac{1}{(2\pi)^d \pi^d} \int_{\mathbb{R}^d} \text{Sinc}(\mathbf{u}) \text{Sinc}(\mathbf{u} + \mathbf{r}\pi) \times \\
&\quad \int_{2\pi[-a/\lambda, a/\lambda]^d} g(\boldsymbol{\omega}) \sum_{j=1}^d \left(f\left(\frac{2u_1}{\lambda} - \omega_1, \dots, \frac{2u_j}{\lambda} - \omega_j, \dots, \frac{2u_d}{\lambda} - \omega_d\right) - f(\boldsymbol{\omega}) \right) d\boldsymbol{\omega} d\mathbf{u}.
\end{aligned}$$

Applying Lemma A.2 to each variable in R_n , gives $|R_n| = O(\Gamma(a/\lambda) \frac{\log \lambda + \log a}{\lambda})$, thus we have the result in the case of $d \geq 2$.

The analysis of $\tilde{Q}_{a,\lambda}(g; \mathbf{r})$ is the same, using the above it is easily seen that

$$E(\tilde{Q}_{a,\lambda}(g; \mathbf{r})) = C_2 \int_{\mathbb{R}^d} \text{Sinc}(\mathbf{u}) \text{Sinc}(\mathbf{u} + \mathbf{r}\pi) \left(\frac{1}{\pi^d \lambda^d} \sum_{\mathbf{k}=-a}^a g(\boldsymbol{\omega}_k) f\left(\frac{2\mathbf{u}}{\lambda} - \boldsymbol{\omega}_k\right) \right) d\mathbf{u}.$$

Thus by using the same method as that given above we have the result. \square

6 The proof of Lemma 3.1 and Theorems 3.2 and 3.3

PROOF of Lemma 3.1 We prove the result in the case $d = 1$ (the proof for $d > 1$ is identical). We first prove (i). By using indecomposable partitions, Theorem 2.1 and Lemma A.6 (in the appendix), noting that the fourth order cumulant is of order $O(1/n)$, it is straightforward to show that

$$\begin{aligned}
&\lambda \text{cov} \left[\tilde{Q}_{a,\lambda}(g; r_1), \tilde{Q}_{a,\lambda}(g; r_2) \right] \\
&= \frac{1}{\lambda} \sum_{k_1, k_2 = -a}^a g(\omega_{k_1}) \overline{g(\omega_{k_2})} [\text{cov}(J_n(\omega_{k_1}) \overline{J_n(\omega_{k_1+r_1})}, J_n(\omega_{k_2}) \overline{J_n(\omega_{k_2+r_2})})] \\
&= \begin{cases} \frac{1}{\lambda} \sum_{k=-a}^a f(\omega_k) f(\omega_k + \omega_r) g(\omega_k) \overline{g(\omega_k)} \\ + \frac{1}{\lambda} \sum_{k=\max(-a, -a-r)}^{\min(a, a-r)} f(\omega_k) f(\omega_k + \omega_r) g(\omega_k) \overline{g(-\omega_{k+r})} + O(\frac{a}{\lambda^2} + \frac{\lambda}{n}) & r_1 = r_2 \\ O(\frac{a}{\lambda^2} + \frac{\lambda}{n}) & r_1 \neq r_2 \end{cases}
\end{aligned}$$

By Assumption 2.4(i), $a = C\lambda$, thus $O(\frac{a}{\lambda^2}) = O(\frac{1}{\lambda})$ and by replacing sum with integral we have

$$\lambda \text{cov} \left[\tilde{Q}_{a,\lambda}(g; r_1), \tilde{Q}_{a,\lambda}(g; r_2) \right] = \begin{cases} C_1(\omega_r) + O(\frac{1}{\lambda} + \frac{\lambda}{n}) & r_1 = r_2 \\ O(\frac{1}{\lambda} + \frac{\lambda}{n}) & r_1 \neq r_2 \end{cases}$$

where

$$C_1(\omega_r) = C_{11}(\omega_r) + C_{12}(\omega_r),$$

with

$$\begin{aligned} C_{11}(\omega_r) &= \frac{1}{2\pi} \int_{-2\pi a/\lambda}^{2\pi a/\lambda} f(\omega) f(\omega + \omega_r) g(\omega) \overline{g(\omega)} d\omega \\ C_{12}(\omega_r) &= \frac{1}{2\pi} \int_{2\pi \max(-a, -a-r)/\lambda}^{2\pi \min(a, a-r)/\lambda} f(\omega) f(\omega + \omega_r) g(\omega) \overline{g(-\omega - \omega_r)} d\omega. \end{aligned}$$

To show that $C_1(\omega_r)$ is real, we note that it is clear that $C_{11}(\omega_r)$ is real. Thus we need only show that $C_{12}(\omega_r)$ is real. To do this, we write $g(\omega) = g_1(\omega) + ig_2(\omega)$, therefore

$$\Im g(\omega) \overline{g(-\omega - \omega_r)} = [g_2(\omega) g_1(-\omega - \omega_r) - g_1(\omega) g_2(-\omega - \omega_r)].$$

Substituting the above into $\Im C_{12}(\omega_r)$ gives us $\Im C_{12}(r) = [C_{121}(r) + C_{122}(r)]$ where

$$\begin{aligned} C_{121}(r) &= \frac{1}{2\pi} \int_{2\pi \max(-a, -a-r)/\lambda}^{2\pi \min(a, a-r)/\lambda} f(\omega) f(\omega + \omega_r) g_2(\omega) g_1(-\omega - \omega_r) d\omega \\ C_{122}(r) &= -\frac{1}{2\pi} \int_{2\pi \max(-a, -a-r)/\lambda}^{2\pi \min(a, a-r)/\lambda} f(\omega) f(\omega + \omega_r) g_1(\omega) g_2(-\omega - \omega_r) d\omega \end{aligned}$$

We will show that $C_{122}(r) = -C_{121}(r)$. Focusing on C_{122} and making the change of variables $u = -\omega - \omega_r$ gives us

$$C_{122} = \frac{1}{2\pi} \int_{2\pi \min(a, a-r)/\lambda}^{2\pi \max(-a, -a-r)/\lambda} f(u + \omega_r) f(-u) g_1(-u - \omega_r) g_2(u) du,$$

noting that the spectral density function is symmetric with $f(-u) = f(u)$, and that $\int_{2\pi \min(a, a-r)/\lambda}^{2\pi \max(-a, -a-r)/\lambda} = -\int_{2\pi \max(-a, -a-r)/\lambda}^{2\pi \min(a, a-r)/\lambda}$. Thus we have $\Im C_{12}(r) = 0$, which shows that $C_1(\omega_r)$ is real. The proof of $\lambda^d \text{cov}[\tilde{Q}_{a,\lambda}(g; \mathbf{r}_1), \overline{\tilde{Q}_{a,\lambda}(g; \mathbf{r}_2)}]$ is the same. Thus, we have proven (i).

To prove (ii) we first expand $\text{cov}[\tilde{Q}_{a,\lambda}(g; r_1), \tilde{Q}_{a,\lambda}(g; r_2)]$ to give

$$\begin{aligned} &\lambda \text{cov}[\tilde{Q}_{a,\lambda}(g; r_1), \tilde{Q}_{a,\lambda}(g; r_2)] \\ &= C_4 \sum_{k_1, k_2 = -a}^a g(\omega_{k_1}) \overline{g(\omega_{k_2})} \sum_{j_1, \dots, j_4 = 1}^n \left(\mathbb{E}[c(s_{j_1} - s_{j_3}) e^{is_1 \omega_{k_1} - is_{j_3} \omega_{k_2}}] \mathbb{E}[c(s_{j_2} - s_{j_4}) e^{-is_2 \omega_{k_1} + r_1 - is_4 \omega_{k_2} + r_2}] + \right. \\ &\quad \mathbb{E}[c(s_{j_1} - s_{j_4}) e^{is_1 \omega_{k_1} - is_4 \omega_{k_2} + r_2}] \mathbb{E}[c(s_{j_2} - s_{j_3}) e^{-is_2 \omega_{k_1} + r_1 + is_3 \omega_{k_2}}] + \\ &\quad \left. \text{cum}[Z(s_{j_1}) e^{i\omega_{k_1} s_{j_1}} Z(s_{j_2}) e^{-is_{j_2}(\omega_{k_1} + \omega_{r_2})}, Z(s_{j_3}) e^{i\omega_{k_2} s_{j_3}}, Z(s_{j_4}) e^{-is_{j_4}(\omega_{k_2} + \omega_{r_2})}] \right), \end{aligned}$$

where $C_4 = n(n-1)(n-2)(n-3)/n^4$. By applying Lemma A.6 to the fourth order cumulant we can show that

$$\lambda \text{cov} \left[\tilde{Q}_{a,\lambda}(r_1), \tilde{Q}_{a,\lambda}(g; r_2) \right] = A_1(r_1, r_2) + A_2(r_1, r_2) + O\left(\frac{\lambda}{n}\right) \quad (23)$$

where

$$\begin{aligned} A_1(r_1, r_2) &= \lambda \sum_{k_1, k_2=-a}^a g(\omega_{k_1}) \overline{g(\omega_{k_2})} \text{cov} \left[Z(s_1) \exp(is_1 \omega_{k_1}), Z(s_3) \exp(is_3 \omega_{k_2}) \right] \times \\ &\quad \text{cov} \left[Z(s_2) \exp(-is_2 \omega_{k_1+r_1}), Z(s_4) \exp(-is_4 \omega_{k_2+r_2}) \right] \\ A_2(r_1, r_2) &= \lambda \sum_{k_1, k_2=-a}^a g(\omega_{k_1}) \overline{g(\omega_{k_2})} \text{cov} \left[Z(s_1) \exp(is_1 \omega_{k_1}), Z(s_4) \exp(-is_4 \omega_{k_2+r_2}) \right] \times \\ &\quad \text{cov} \left[Z(s_2) \exp(-is_2 \omega_{k_1+r_1}), Z(s_3) \exp(is_3 \omega_{k_2}) \right]. \end{aligned}$$

Note we have cut of corners by replacing $n(n-1)(n-2)(n-3)/n^4 = 1 + O(n^{-1})$ (because, later in the proof, we will show that $A_1(r_1, r_2)$ and $A_2(r_1, r_2)$ are both bounded over λ and a). To write $A_1(r_1, r_2)$ in the form stated in the lemma we condition on s_1, \dots, s_4 to give

$$\begin{aligned} &A_1(r_1, r_2) \\ &= \lambda \sum_{k_1, k_2=-a}^a g(\omega_{k_1}) \overline{g(\omega_{k_2})} \frac{1}{\lambda^4} \int_{[-\lambda/2, \lambda/2]^4} c(s_1 - s_3) c(s_2 - s_4) e^{i(s_1 \omega_{k_1} - s_3 \omega_{k_2})} e^{-is_2 \omega_{k_1+r_1} + is_4 \omega_{k_2+r_2}} ds_1 \dots ds_4. \end{aligned}$$

By using the spectral representation theorem and integrating out s_1, \dots, s_4 we can write the above as

$$\begin{aligned} &A_1(r_1, r_2) \\ &= \frac{\lambda}{(2\pi)^2} \sum_{k_1, k_2=-a}^a g(\omega_{k_1}) \overline{g(\omega_{k_2})} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) f(y) \text{sinc}\left(\frac{\lambda x}{2} + k_1 \pi\right) \\ &\quad \text{sinc}\left(\frac{\lambda y}{2} - (r_1 + k_1) \pi\right) \text{sinc}\left(\frac{\lambda x}{2} + k_2 \pi\right) \text{sinc}\left(\frac{\lambda y}{2} - (r_2 + k_2) \pi\right) dx dy \\ &= \frac{1}{\pi^2 \lambda} \sum_{k_1, k_2=-a}^a g(\omega_{k_1}) \overline{g(\omega_{k_2})} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f\left(\frac{2u}{\lambda} - \omega_{k_1}\right) f\left(\frac{2v}{\lambda} + \omega_{k_1} + \omega_{r_1}\right) \\ &\quad \times \text{sinc}(u) \text{sinc}(u + (k_2 - k_1) \pi) \text{sinc}(v) \text{sinc}(v + (k_1 - k_2 + r_1 - r_2) \pi) du dv, \end{aligned}$$

where the second equality is due to the change of variables $u = \frac{\lambda x}{2} + k_1 \pi$ and $v = \frac{\lambda y}{2} - (r_1 + k_1) \pi$. Finally, by making a change of variables $m = k_1 - k_2$ we obtain the expression for $A_1(r_1, r_2)$ given in Lemma 3.1.

A similar method can be used to obtain the expression for $A_2(r_1, r_2)$. Finally following the same steps as those above we obtain

$$\lambda \text{cov}(\tilde{Q}_{a,\lambda}(g; r_1), \overline{\tilde{Q}_{a,\lambda}(g; r_2)}) = A_3(r_1, r_2) + A_4(r_1, r_2) + O\left(\frac{\lambda}{n}\right),$$

where

$$\begin{aligned}
A_3(r_1, r_2) &= \frac{1}{\lambda^4} \sum_{k_1, k_2 = -n}^n g(\omega_{k_1}) g(\omega_{k_2}) \int_{[-\lambda/2, \lambda/2]^4} e^{i\omega_{k_1}(s_1-s_2)} e^{-is_2\omega_{r_1} - is_4\omega_{r_2}} e^{i\omega_{k_2}(s_3-s_4)} \\
&\quad \times c(s_1, s_3) c(s_2, s_4) ds_1 ds_2 ds_3 ds_4 \\
A_4(r_1, r_2) &= \frac{1}{\lambda^4} \sum_{k_1, k_2 = -n}^n g(\omega_{k_1}) g(\omega_{k_2}) \int_{[-\lambda/2, \lambda/2]^4} e^{i\omega_{k_1}(s_1-s_2)} e^{-is_2\omega_{r_1} - is_4\omega_{r_2}} e^{i\omega_{k_2}(s_3-s_4)} \\
&\quad \times c(s_1, s_4) c(s_2, s_3) ds_1 ds_2 ds_3 ds_4.
\end{aligned}$$

Again by replacing the covariances in $A_3(r_1, r_2)$ and $A_4(r_1, r_4)$ with their spectral representation gives (ii) for $d = 1$. The result for $d > 1$ is identical.

It is clear that (iii) is true under Assumption 2.4(i)(a,b). To prove (iii) under Assumption 2.4(ii)(a,b) we will show that for $1 \leq j \leq 4$, $\sup_a |A_j(\mathbf{r}_1, \mathbf{r}_2)| < \infty$. To do this, we first note that by the Cauchy Schwarz inequality we have

$$\begin{aligned}
&\sup_{a, \lambda} \frac{1}{\pi^{2d} \lambda^d} \left| \sum_{\mathbf{k} = \max(-a, -a+\mathbf{m})}^{\min(-a+\mathbf{m})} g(\omega_{\mathbf{k}}) \overline{g(\omega_{\mathbf{k}} - \omega_{\mathbf{m}})} f\left(\frac{2\mathbf{u}}{\lambda} - \omega_{\mathbf{k}}\right) f\left(\frac{2\mathbf{v}}{\lambda} + \omega_{\mathbf{k}} + \omega_{\mathbf{r}_1}\right) \right| \\
&\leq C \sup_{\omega} |g(\omega)|^2 \|f\|_2^2,
\end{aligned}$$

where $\|f\|_2$ is the ℓ_2 norm of the spectral density function and C is a finite constant. Thus by taking absolutes of $A_1(\mathbf{r}_1, \mathbf{r}_2)$ we have

$$\begin{aligned}
&|A_1(\mathbf{r}_1, \mathbf{r}_2)| \leq \\
&C \|f\|_2^2 \sup_{\omega} |g(\omega)|^2 \sum_{\mathbf{m} = -\infty}^{\infty} \int_{\mathbb{R}^{2d}} |\text{Sinc}(\mathbf{u} + \mathbf{m}\pi) \text{Sinc}(\mathbf{v} - (\mathbf{m} + \mathbf{r}_1 - \mathbf{r}_2)\pi) \text{Sinc}(\mathbf{u}) \text{Sinc}(\mathbf{v})| d\mathbf{u} d\mathbf{v}.
\end{aligned}$$

Finally, by using Lemma A.1(iii) we have that $\sup_a |A_1(\mathbf{r}_1, \mathbf{r}_2)| < \infty$. By using the same method we can show that $\sup_a |A_1(\mathbf{r}_1, \mathbf{r}_2)|, \dots, \sup_a |A_4(\mathbf{r}_1, \mathbf{r}_4)| < \infty$. This completes the proof. \square

Lemma 6.1 *Suppose Assumptions 2.1 and 2.4(ii)(a,b) hold. Then we have*

$$\begin{aligned}
A_1(r_1, r_2) &= \begin{cases} C_1(\omega_r) + O(\ell_{\lambda, a, n}) & r_1 = r_2 = r \\ O(\ell_{\lambda, a, n}) & r_1 \neq r_2 \end{cases} \\
A_2(r_1, r_2) &= \begin{cases} C_2(\omega_r) + O(\ell_{\lambda, a, n}) & r_1 = r_2 (= r) \\ O(\ell_{\lambda, a, n}) & r_1 \neq r_2 \end{cases}
\end{aligned}$$

$$A_3(r_1, r_2) = \begin{cases} C_3(\omega_r) + O(\ell_{\lambda,a,n}) & r_1 = -r_2 (= r) \\ O(\ell_{\lambda,a,n}) & r_1 \neq -r_2 \end{cases}$$

and

$$A_4(r_1, r_2) = \begin{cases} C_4(\omega_r) + O(\ell_{\lambda,a,n}) & r_1 = -r_2 (= r) \\ O(\ell_{\lambda,a,n}) & r_1 \neq -r_2 \end{cases}.$$

where $C_j(\omega_r)$ (using $d = 1$) and $\ell_{\lambda,a,n}$ are defined in Theorem 3.2.

PROOF. We write $A_1(r_1, r_2)$ as

$$A_1(r_1, r_2) = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{m=-2a}^{2a} \text{sinc}(u) \text{sinc}(u + (m + r_1 - r_2)\pi) \text{sinc}(v) \text{sinc}(v + m\pi) H_{m,\lambda}\left(\frac{2u}{\lambda}, \frac{2v}{\lambda}; r_1\right) dudv,$$

where

$$H_{m,\lambda}\left(\frac{2u}{\lambda}, \frac{2v}{\lambda}; r_1\right) = \frac{1}{\lambda} \sum_{k=\max(-a, -a+m)}^{\min(a, a+m)} f\left(-\frac{2u}{\lambda} + \omega_k\right) f\left(\frac{2v}{\lambda} + \omega_k + \omega_r\right) g(\omega_k) \overline{g(\omega_k + \omega_m)}$$

noting that $f(\frac{2u}{\lambda} - \omega) = f(\omega - \frac{2u}{\lambda})$.

If $f(-\frac{2u}{\lambda} + \omega_k)$ and $f(\frac{2v}{\lambda} + \omega_k + \omega_r)$ we replaced with $f(\omega_k)$ and $f(\omega_k + \omega_r)$ respectively, then we can exploit the orthogonality property of the sinc functions. This requires the assumptions given in the lemma and the following series of approximations.

- (i) We start by defining a similar version of $A_1(r_1, r_2)$ but with the sum replaced with an integral. Let

$$B_1(r_1 - r_2; r_1) = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{m=-2a}^{2a} \text{sinc}(u) \text{sinc}(u + m\pi) \text{sinc}(v) \text{sinc}(v + (m + r_1 - r_2)\pi) H_m\left(\frac{2u}{\lambda}, \frac{2v}{\lambda}; r_1\right) dudv,$$

where

$$H_m\left(\frac{2u}{\lambda}, \frac{2v}{\lambda}; r_1\right) = \frac{1}{2\pi} \int_{2\pi \max(-a, -a+m)/\lambda}^{2\pi \min(a, a+m)/\lambda} f\left(\omega - \frac{2u}{\lambda}\right) f\left(\frac{2v}{\lambda} + \omega + \omega_{r_1}\right) g(\omega) \overline{g(\omega + \omega_m)} d\omega.$$

In Lemma A.4 we show that

$$|A_1(r_1, r_2) - B_1(r_1 - r_2; r_1)| = O\left(\frac{\log^2(a)}{\lambda}\right).$$

(ii) Define the quantity

$$C_1(r_1 - r_2; r_1) = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{m=-2a}^{2a} \text{sinc}(u) \text{sinc}(u + m\pi) \text{sinc}(v) \text{sinc}(v + (m + r_1 - r_2)\pi) H_m(0, 0; r_1) du dv,$$

We show in Lemma A.5 that $B_1(r_1 - r_2; r_1)$ can be replaced with $C_1(r_1 - r_2; r_1)$ and

$$|B_1(r_1 - r_2; r_1) - C_1(r_1 - r_2; r_1)| = O\left(\log^2(a) \left[\Gamma(a/\lambda) \frac{(\log a + \log \lambda)}{\lambda} \right]\right).$$

(iii) Finally, we analyze $C_1(r_1 - r_2; r_1)$. Since $H_m(0, 0; r_1)$ does not depend on u or v we have

$$\begin{aligned} C_1(r_1 - r_2; r_1) &= \frac{1}{\pi^2} \sum_{m=-2a}^{2a} H_m(0, 0; r_1) \left(\int_{\mathbb{R}} \text{sinc}(u) \text{sinc}(u + m\pi) du \right) \left(\int_{\mathbb{R}} \text{sinc}(v) \text{sinc}(v + (m + r_1 - r_2)\pi) dv \right) \\ &= \frac{1}{\pi^2} H_0(0, 0; r_1) \left(\int_{\mathbb{R}} \text{sinc}^2(u) du \right) \left(\int_{\mathbb{R}} \text{sinc}(v) \text{sinc}(v + (r_1 - r_2)\pi) dv \right), \end{aligned}$$

where the above line is due to Lemma 5.1. If $r_1 \neq r_2$, then by Lemma 5.1 $C_1(r_1 - r_2; r_1) = 0$. On the other hand if $r_1 = r_2$, then again by Lemma 5.1, we have

$$C_1(r_1 - r_2; r_1) = \int_{-a/\lambda}^{a/\lambda} f(\omega) f(\omega + \omega_{r_1}) g(\omega) g(\omega + \omega) d\omega = C_1(\omega_r).$$

The proof for the remaining terms $A_2(r_1, r_2)$, $A_3(r_1, r_2)$ and $A_4(r_1, r_2)$ is identical, thus we omit the details. \square

PROOF of Theorem 3.2. By using Lemmas 3.1 and 6.1 we immediately obtain (in the case $d = 1$)

$$\text{cov} \left[\tilde{Q}_{a,\lambda}(g; r_1), \tilde{Q}_{a,\lambda}(g; r_2) \right] = \begin{cases} C_1(\omega_r) + C_2(\omega_r) + O(\ell_{\lambda,a,n}) & r_1 = r_2 \\ O(\ell_{\lambda,a,n}) & r_1 \neq r_2 \end{cases}$$

and

$$\text{cov} \left[\tilde{Q}_{a,\lambda}(g; r_1), \overline{\tilde{Q}_{a,\lambda}(g; r_2)} \right] = \begin{cases} C_3(\omega_r) + C_4(\omega_r) + O(\ell_{\lambda,a,n}) & r_1 = -r_2 \\ O(\ell_{\lambda,a,n}) & r_1 \neq -r_2 \end{cases}$$

This gives the result for $d = 1$. To prove the result for $d > 1$ we use the same procedure outlined in the proof of Lemma 6.1 and the above. \square

PROOF of Lemma 3.2 By Lipschitz continuity of $g(\cdot)$ and $f(\cdot)$, it is straightforward to show $C_j(\omega_r) = C_j + O(\frac{\|r\|_1}{\lambda})$. \square

PROOF of Theorem 3.3. We prove the result for the notationally simple case $d = 1$. By using indecomposable partitions, conditional cumulants and (23) we have

$$\begin{aligned} & \lambda \text{cov} \left[\tilde{Q}_{a,\lambda}(r_1), \tilde{Q}_{a,\lambda}(g; r_2) \right] \\ &= A_1(r_1, r_2) + A_2(r_1, r_2) + B_1(r_1, r_2) + B_2(r_1, r_2) + O\left(\frac{\lambda}{n}\right) \end{aligned} \quad (24)$$

where $A_1(r_1, r_2)$ and $A_2(r_1, r_2)$ are defined below equation (23) and

$$\begin{aligned} B_1(r_1, r_2) &= \lambda C_4 \sum_{k_1, k_2 = -a}^a g(\omega_{k_1}) \overline{g(\omega_{k_2})} \mathbb{E} \left[\kappa_4(s_1 - s_2, s_1 - s_3, s_1 - s_4) e^{is_1 \omega_{k_1}} e^{-is_2 \omega_{k_1} + r_1} e^{-is_3 \omega_{k_2}} e^{is_4 \omega_{k_2} + r_2} \right] \\ B_2(r_1, r_2) &= \frac{\lambda}{n^4} \sum_{j_1, \dots, j_4 \in \mathcal{D}_3} \sum_{k_1, k_2 = -a}^a g(\omega_{k_1}) \overline{g(\omega_{k_2})} \times \\ & \quad \mathbb{E} \left[\kappa_4(s_{j_1} - s_{j_2}, s_{j_1} - s_{j_3}, s_{j_1} - s_{j_4}) e^{is_{j_1} \omega_{k_1}} e^{-is_{j_2} \omega_{k_1} + r_1} e^{-is_{j_3} \omega_{k_2}} e^{is_{j_4} \omega_{k_2} + r_2} \right] \end{aligned}$$

with $\mathcal{D}_3 = \{j_1, \dots, j_4; j_1 \neq j_2 \text{ and } j_3 \neq j_4 \text{ but some } j\text{'s are in common}\}$. The limits of $A_1(r_1, r_2)$ and $A_2(r_1, r_2)$ are given in Theorem 3.2, therefore, all that remains is to derive bounds for $B_1(r_1, r_2)$ and $B_2(r_1, r_2)$. We will show that $B_1(r_1, r_2)$ is the dominating term, whereas by placing sufficient conditions on the rate of growth of a , we will show that $B_2(r_1, r_2) \rightarrow 0$. Substituting the Fourier representation

$$\kappa_4(s_1 - s_2, s_1 - s_3, s_1 - s_4) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} f_4(\omega_1, \omega_2, \omega_3) e^{i(s_1 - s_2)\omega_1} e^{i(s_1 - s_3)\omega_2} e^{i(s_1 - s_4)\omega_4} d\omega_1 d\omega_2 d\omega_3,$$

into $B_1(r_1, r_2)$ we have

$$\begin{aligned} & B_1(r_1, r_2) \\ &= \frac{C_4}{(2\pi)^3 \lambda^3} \sum_{k_1, k_2 = -a}^a g(\omega_{k_1}) \overline{g(\omega_{k_2})} \int_{\mathbb{R}^3} f_4(\omega_1, \omega_2, \omega_3) \int_{[-\lambda/2, \lambda/2]^4} e^{is_1(\omega_1 + \omega_2 + \omega_3 + \omega_{k_1})} \\ & \quad e^{-is_2(\omega_1 + \omega_{k_1} + r_1)} e^{-is_3(\omega_2 + \omega_{k_2})} e^{is_4(-\omega_3 + \omega_{k_2} + r_2)} ds_1 ds_2 ds_3 ds_4 d\omega_1 d\omega_2 d\omega_3 \\ &= \frac{C_4 \lambda}{(2\pi)^3} \sum_{k_1, k_2 = -a}^a g(\omega_{k_1}) \overline{g(\omega_{k_2})} \int_{\mathbb{R}^3} f_4(\omega_1, \omega_2, \omega_3) \text{sinc}\left(\frac{\lambda(\omega_1 + \omega_2 + \omega_3)}{2} + k_1 \pi\right) \text{sinc}\left(\frac{\lambda \omega_1}{2} + (k_1 + r_1) \pi\right) \\ & \quad \times \text{sinc}\left(\frac{\lambda \omega_2}{2} + k_2 \pi\right) \text{sinc}\left(\frac{\lambda \omega_3}{2} - (k_2 + r_2) \pi\right) d\omega_1 d\omega_2 d\omega_3. \end{aligned}$$

Now we make a change of variable and let $u_1 = \frac{\lambda \omega_1}{2} + (k_1 + r_1)$, $u_2 = \frac{\lambda \omega_2}{2} + k_2 \pi$ and

$u_3 = \frac{\lambda\omega_3}{2} - (k_2 + r_2)\pi$, this gives

$$B_1(r_1, r_2) = \frac{C_4}{(\pi)^3 \lambda^2} \sum_{k_1, k_2 = -a}^a \int_{\mathbb{R}^3} g(\omega_{k_1}) \overline{g(\omega_{k_2})} f_4\left(\frac{2u_1}{\lambda} - \omega_{k_1-r_1}, \frac{2u_2}{\lambda} - \omega_{k_2}, \frac{2u_3}{\lambda} + \omega_{k_2+r_2}\right) \times \\ \times \text{sinc}(u_1 + u_2 + u_3 + (r_1 - r_2)\pi) \text{sinc}(u_1) \text{sinc}(u_2) \text{sinc}(u_3) du_1 du_2 du_3.$$

By using Lemma A.3, we replace $f_4(\frac{2u_1}{\lambda} - \omega_{k_1-r_1}, \frac{2u_2}{\lambda} - \omega_{k_2}, \frac{2u_3}{\lambda} + \omega_{k_2+r_2})$ in the integral with $f_4(-\omega_{k_1-r_1}, -\omega_{k_2}, \omega_{k_2+r_2})$, this gives

$$B_1(r_1, r_2) = \frac{C_4}{(\pi)^3} \int_{[-a/\lambda, a/\lambda]^2} g(\omega_1) \overline{g(\omega_2)} f_4(-\omega_{k_1} + \omega_{r_1}, -\omega_{k_2}, \omega_{k_2} + \omega_{r_2}) \times \\ \int_{\mathbb{R}^3} \text{sinc}(u_1 + u_2 + u_3 + (r_1 - r_2)\pi) \text{sinc}(u_1) \text{sinc}(u_2) \text{sinc}(u_3) du_1 du_2 du_3 + O\left(\frac{\log^3 \lambda}{\lambda}\right) \\ = \frac{C_4}{(\pi)^3} \int_{[-a/\lambda, a/\lambda]^2} g(\omega_1) \overline{g(\omega_2)} f_4(-\omega_1, -\omega_2, \omega_2) d\omega_1 d\omega_2 \times \\ \int_{\mathbb{R}^3} \text{sinc}(u_1 + u_2 + u_3 + (r_1 - r_2)\pi) \text{sinc}(u_1) \text{sinc}(u_2) \text{sinc}(u_3) du_1 du_2 du_3 + O\left(\frac{\log^3 \lambda}{\lambda} + \frac{|r_1| + |r_2|}{\lambda}\right).$$

By using Lemma 5.1 we can show that

$$\int_{\mathbb{R}^3} \text{sinc}(u_1 + u_2 + u_3 + (r_1 - r_2)\pi) \text{sinc}(u_2) \text{sinc}(u_3) du_1 du_2 du_3 = \begin{cases} \pi^3 & r_1 = r_2 \\ 0 & r_1 \neq r_2. \end{cases}$$

Using the above identities gives

$$B_1(r_1, r_2) = \begin{cases} \int_{\mathbb{R}^3} \int_{[-a/\lambda, a/\lambda]^2} g(\omega_1) \overline{g(\omega_2)} f_4(-\omega_1, -\omega_2, \omega_2) d\omega_1 d\omega_2 + O\left(\frac{\log^3 \lambda}{\lambda} + \frac{|r_1| + |r_2|}{\lambda}\right) & r_1 = r_2 \\ O\left(\frac{\log^3 \lambda}{\lambda} + \frac{|r_1| + |r_2|}{\lambda}\right) & r_1 \neq r_2. \end{cases}$$

To bound $B_2(r_1, r_2)$ we decompose \mathcal{D}_3 into six sets, the set $\mathcal{D}_{3,1} = \{j_1, \dots, j_4; j_1 = j_3, j_2 \text{ and } j_4 \text{ are different}\}$, $\mathcal{D}_{3,2} = \{j_1, \dots, j_4; j_1 = j_4, j_2 \text{ and } j_3 \text{ are different}\}$, $\mathcal{D}_{3,3} = \{j_1, \dots, j_4; j_2 = j_3, j_1 \text{ and } j_4 \text{ are different}\}$, $\mathcal{D}_{3,4} = \{j_1, \dots, j_4; j_2 = j_4, j_1 \text{ and } j_3 \text{ are different}\}$, $\mathcal{D}_{2,1} = \{j_1, \dots, j_4; j_1 = j_3 \text{ and } j_2 = j_4\}$, $\mathcal{D}_{2,2} = \{j_1, \dots, j_4; j_1 = j_4 \text{ and } j_2 = j_3\}$. Using this we have $B_2(r_1, r_2) = \sum_{j=1}^4 B_{2,(3,j)}(r_1, r_2) + \sum_{j=1}^2 B_{2,(2,j)}(r_1, r_2)$, where

$$B_{2,(3,1)}(r_1, r_2) = \frac{|\mathcal{D}_{3,1}| \lambda}{n^4} \sum_{k_1, k_2 = -a}^a g(\omega_{k_1}) \overline{g(\omega_{k_2})} \times \\ \mathbb{E} \left[\kappa_4(s_{j_2} - s_{j_1}, 0, s_{j_4} - s_{j_1}) e^{is_{j_1} \omega_{k_1}} e^{-is_{j_2} \omega_{k_1+r_1}} e^{-is_{j_1} \omega_{k_2}} e^{is_{j_4} \omega_{k_2+r_2}} \right]$$

$B_{2,(3,j)}(r_1, r_2)$ (for $j = 2, 3, 4$) are defined similarly,

$$B_{2,(2,1)}(r_1, r_2) = \frac{|\mathcal{D}_{2,1}|\lambda}{n^4} \sum_{k_1, k_2 = -a}^a g(\omega_{k_1}) \overline{g(\omega_{k_2})} \times \\ \mathbb{E} \left[\kappa_4(s_{j_2} - s_{j_1}, 0, s_{j_2} - s_{j_1}) e^{is_{j_1}\omega_{k_1}} e^{-is_{j_2}\omega_{k_1}+r_1} e^{-is_{j_1}\omega_{k_2}} e^{is_{j_2}\omega_{k_2}+r_2} \right],$$

$B_{2,(2,2)}(r_1, r_2)$ is defined similarly and $|\cdot|$ denotes the cardinality of a set. By using identical methods to those used to bound $B_1(r_1, r_2)$ we have

$$|B_{2,(3,1)}(r_1, r_2)| \\ \leq \frac{C\lambda}{n(2\pi)^3} \sum_{k_1, k_2 = -a}^a |g(\omega_{k_1}) \overline{g(\omega_{k_2})}| \int_{\mathbb{R}^3} |f_4(\omega_1, \omega_2, \omega_3)| \left| \text{sinc}\left(\frac{\lambda(\omega_1 + \omega_3)}{2} + (k_1 - k_2)\pi\right) \right| \times \\ \left| \text{sinc}\left(\frac{\lambda\omega_1}{2} + (k_1 + r_1)\pi\right) \right| \times \left| \text{sinc}\left(\frac{\lambda\omega_3}{2} - (k_2 + r_2)\pi\right) \right| d\omega_1 d\omega_2 d\omega_3 = O\left(\frac{\lambda}{n}\right).$$

Similarly we can show $|B_{2,(3,j)}(r_1, r_2)| = O(\frac{\lambda}{n})$ (for $2 \leq j \leq 4$) and

$$|B_{2,(2,1)}(r_1, r_2)| \leq \frac{\lambda}{n^2} \sum_{k_1, k_2 = -a}^a |g(\omega_{k_1}) \overline{g(\omega_{k_2})}| \int_{\mathbb{R}^3} |f_4(\omega_1, \omega_2, \omega_3)| \times \\ \left| \text{sinc}\left(\frac{\lambda(\omega_1 + \omega_2)}{2} + (k_1 - k_2)\pi\right) \text{sinc}\left(\frac{\lambda(\omega_1 + \omega_2)}{2} + (k_1 - k_2 + r_1 - r_2)\pi\right) \right| d\omega_1 d\omega_2 \\ = O\left(\frac{a\lambda}{n^2}\right).$$

Thus if $a = O(n)$, then $|B_2(r_1, r_2)| = O(\frac{\lambda}{n})$. This immediately gives us (14). To prove (15) we use identical methods. Thus we obtain the result. \square

PROOF of Theorem 3.5 By using the well known identities

$$\begin{aligned} \text{cov}(\Re A, \Re B) &= \frac{1}{2} (\Re \text{cov}(A, B) + \Re \text{cov}(A, \bar{B})) \\ \text{cov}(\Im A, \Im B) &= \frac{1}{2} (\Re \text{cov}(A, B) - \Re \text{cov}(A, \bar{B})), \\ \text{cov}(\Re A, \Im B) &= \frac{-1}{2} (\Im \text{cov}(A, B) - \Im \text{cov}(A, \bar{B})), \end{aligned} \tag{25}$$

and Lemma 3.2 we immediately obtain

$$\lambda^d \text{cov} \left[\Re \tilde{Q}_{a,\lambda}(g; \mathbf{r}_1), \Re \tilde{Q}_{a,\lambda}(g; \mathbf{r}_2) \right] = \begin{cases} \frac{\Re}{2} C_1(\boldsymbol{\omega}_{\mathbf{r}}) + O(\ell_{\lambda,a,n}) & \mathbf{r}_1 = \mathbf{r}_2 (= \mathbf{r}) \\ \frac{\Re}{2} C_2(\boldsymbol{\omega}_{\mathbf{r}}) + O(\ell_{\lambda,a,n}) & \mathbf{r}_1 = -\mathbf{r}_2 (= \mathbf{r}) \\ O(\ell_{\lambda,a,n}) & \text{otherwise} \end{cases}$$

$$\lambda^d \text{cov} \left[\Im \tilde{Q}_{a,\lambda}(g; \mathbf{r}_1), \Im \tilde{Q}_{a,\lambda}(g; \mathbf{r}_2) \right] = \begin{cases} \frac{\Re}{2} C_1(\boldsymbol{\omega}_{\mathbf{r}}) + O(\ell_{\lambda,a,n}) & \mathbf{r}_1 = \mathbf{r}_2 (= \mathbf{r}) \\ \frac{-\Re}{2} C_2(\boldsymbol{\omega}_{\mathbf{r}}) + O(\ell_{\lambda,a,n}) & \mathbf{r}_1 = -\mathbf{r}_2 (= \mathbf{r}) \\ O(\ell_{\lambda,a,n}) & \text{otherwise} \end{cases}$$

and

$$\lambda^d \text{cov} \left[\Re \tilde{Q}_{a,\lambda}(g; \mathbf{r}_1), \Im \tilde{Q}_{a,\lambda}(g; \mathbf{r}_2) \right] = \begin{cases} O(\ell_{\lambda,a,n}) & \mathbf{r}_1 \neq -\mathbf{r}_2 \\ \frac{\Im}{2} C_2(\boldsymbol{\omega}_{\mathbf{r}}) + O(\ell_{\lambda,a,n}) & \mathbf{r}_1 = -\mathbf{r}_2 (= \mathbf{r}) \end{cases}$$

Finally, asymptotic normality of $\Re \tilde{Q}_{a,\lambda}(g; \mathbf{r})$ and $\Im \tilde{Q}_{a,\lambda}(g; \mathbf{r})$ follows from Theorem 3.4. Thus we obtain the result. \square

7 Proofs in the case of non-uniform sampling

In this section we prove the results in Section 4. Most of the results derived here are based on the methodology developed in the uniform sampling case.

PROOF of Lemma 2.2 We prove the result for $d = 1$. From (17) in Section 4 we have

$$\begin{aligned} \text{cov} [J_T(\omega_{k_1}), J_T(\omega_{k_2})] &= \frac{C_2}{\lambda} \sum_{j_1, j_2 = -\infty}^{\infty} \gamma_{j_1} \gamma_{j_2} \int_{-\lambda/2}^{\lambda/2} \int_{-\lambda/2}^{\lambda/2} c(s_1 - s_2) e^{2\pi i \frac{s_1 j_1}{\lambda}} e^{2\pi i \frac{j_2 s_2}{\lambda}} e^{is_1 \omega_{k_1} - is_2 \omega_{k_2}} ds_1 ds_2 \\ &\quad + \frac{c(0) \gamma_{k_1 - k_2} \lambda}{n}. \end{aligned}$$

Thus, by using the same arguments as those used in the proof of Theorem 2.1, Bandyopadhyay and Subba Rao (2014), we can show that

$$\begin{aligned} \text{cov} [J_T(\omega_{k_1}), J_T(\omega_{k_2})] &= \frac{C_2}{\lambda} \sum_{j_1, j_2 = -\infty}^{\infty} \gamma_{j_1} \gamma_{j_2} \int_{-\lambda/2}^{\lambda/2} e^{is(\omega_{k_2} - \omega_{j_2} - \omega_{j_1} - \omega_{k_1})} \int_{-\lambda/2}^{\lambda/2} c(t) e^{it(\omega_{j_1} + \omega_{k_2})} dt ds \\ &\quad + \frac{c(0) \gamma_{k_1 - k_2} \lambda}{n} + O\left(\frac{1}{\lambda}\right), \end{aligned}$$

where to obtain the remainder $O(\frac{1}{\lambda})$ we use that $\sum_{j=-\infty}^{\infty} |\gamma_j| < \infty$. Next, by using the identity $\int_{-\lambda/2}^{\lambda/2} e^{is(\omega_{k_2} - \omega_{j_2} - \omega_{j_1} - \omega_{k_1})} ds = 0$ unless $k_2 - j_2 = k_1 + j_1$ we have

$$\begin{aligned} \text{cov} [J_T(\omega_{k_1}), J_T(\omega_{k_2})] &= C_2 \sum_{j=-\infty}^{\infty} \gamma_j \gamma_{k_2 - k_1 - j} \int_{-\lambda/2}^{\lambda/2} c(t) e^{it(\omega_{j_1} + \omega_{k_2})} dt ds + \frac{c(0) \gamma_{k_1 - k_2} \lambda}{n} + O\left(\frac{1}{\lambda}\right) \\ &= \sum_{j=-\infty}^{\infty} \gamma_j \gamma_{k_2 - k_1 - j} f(\omega_{k_2 + j}) + \frac{c(0) \gamma_{k_1 - k_2} \lambda}{n} + O\left(\frac{1}{\lambda} + \frac{1}{n}\right). \end{aligned}$$

Finally, we replace $f(\omega_{k_2+j})$ with $f(\omega_{k_2})$ and use the Lipschitz continuity of $f(\cdot)$ to give

$$\left| \sum_{j=-\infty}^{\infty} \gamma_j \gamma_{k_2-k_1-j} [f(\omega_{k_2}) - f(\omega_{k_2+j})] \right| \leq \frac{C}{\lambda} \sum_{j=-\infty}^{\infty} |j| \cdot |\gamma_j \gamma_{k_2-k_1-j}| = O\left(\frac{1}{\lambda}\right).$$

Altogether, this gives

$$\text{cov}[J_T(\omega_{k_1}), J_T(\omega_{k_2})] = f(\omega_{k_2}) \sum_{j=-\infty}^{\infty} \gamma_j \gamma_{k_2-k_1-j} + \frac{c(0)\gamma_{k_1-k_2}\lambda}{n} + O\left(\frac{1}{\lambda} + \frac{1}{n}\right).$$

This complete the proof for $d = 1$, the proof for $d > 1$ is the same. \square

PROOF of Theorem 4.1 The proof of (i) follows immediately from Lemma 2.2, thus we omit the details.

To prove (ii) we use the same method used to prove Lemma 3.1 to obtain

$$\begin{aligned} & \mathbb{E}[\tilde{Q}_{a,\lambda}(g; r)] \\ &= \frac{C_2}{2\pi} \sum_{j_1, j_2=-\infty}^{\infty} \gamma_{j_1} \gamma_{j_2} \sum_{k=-a}^a g(\omega_k) \int_{-\infty}^{\infty} f(\omega) \text{sinc}\left(\frac{\lambda\omega}{2} + (k + j_1)\pi\right) \text{sinc}\left(\frac{\lambda\omega}{2} + (k + r - j_2)\pi\right) d\omega. \end{aligned}$$

By the change of variables $y = \frac{\lambda\omega}{2} + (k + j_1)\pi$ we obtain

$$\begin{aligned} & \mathbb{E}(\tilde{Q}_{a,\lambda}(g; r)) \\ &= \frac{C_2}{\lambda\pi} \sum_{j_1, j_2=-\infty}^{\infty} \gamma_{j_1} \gamma_{j_2} \sum_{k=-a}^a g(\omega_k) \int_{-\infty}^{\infty} f\left(\frac{2y}{\lambda} - \omega_k\right) \text{sinc}(y) \text{sinc}(y + (r - j_1 - j_2)\pi) dy d\omega. \end{aligned}$$

Replacing $f(\frac{2y}{\lambda} - \omega_k)$ and using Lemma A.2 we have

$$\begin{aligned} & \mathbb{E}(\tilde{Q}_{a,\lambda}(g; r)) \\ &= \frac{C_2}{\pi} \sum_{j_1, j_2=-\infty}^{\infty} \gamma_{j_1} \gamma_{j_2} \int_{-a/\lambda}^{a/\lambda} g(\omega) f(-\omega) d\omega \int_{-\infty}^{\infty} \text{sinc}(y) \text{sinc}(y + (r - j_1 - j_2)\pi) dy d\omega + \\ & \quad O\left(\frac{\Gamma(a/\lambda)(\log \lambda + \log a)}{\lambda}\right), \end{aligned}$$

and by the orthogonality of the sinc function at integer shifts (and $f(-\omega) = f(\omega)$) we have

$$\begin{aligned} & \mathbb{E}(\tilde{Q}_{a,\lambda}(g; r)) \\ &= \sum_{j=-\infty}^{\infty} \gamma_j \gamma_{r-j} \int_{-a/\lambda}^{a/\lambda} g(\omega) f(\omega) d\omega + O\left(\frac{\Gamma(a/\lambda)(\log \lambda + \log a)}{\lambda}\right), \end{aligned}$$

thus we obtain the desired result. \square

PROOF of Theorem 4.1 To prove (i) we use Lemmas 2.2 and A.6 to give

$$\lambda \text{cov}(\tilde{Q}_{a,\lambda}(g; r_1), \tilde{Q}_{a,\lambda}(g; r_2)) = A_1(r_1, r_2) + A_2(r_1, r_2) + O\left(\frac{\lambda}{n} + \frac{1}{\lambda}\right)$$

where

$$A_1(r_1, r_2) = \frac{1}{\lambda} \sum_{k_1, k_2 = -a}^a g(\omega_{k_1}) \overline{g(\omega_{k_2})} \left(f(\omega_{k_2}) \sum_{j_1} \gamma_{j_1} \gamma_{k_2 - k_1 - j_1} \right) \left(f(\omega_{k_2 + r_2}) \sum_{j_2} \gamma_{j_2} \gamma_{k_2 - k_1 + r_2 - r_1 - j_2} \right)$$

and

$$A_2(r_1, r_2) = \frac{1}{\lambda} \sum_{k_1, k_2 = -a}^a g(\omega_{k_1}) \overline{g(\omega_{k_2})} \left(f(\omega_{k_2 + r_2}) \sum_{j_1} \gamma_{j_1} \gamma_{k_2 + k_1 + r_2 - j_1} \right) \left(f(\omega_{k_2}) \sum_{j_2} \gamma_{j_2} \gamma_{k_2 + k_1 + r_1 - j_2} \right).$$

We start by analysing $A_1(r_1, r_2)$. We change variables with $m = k_2 - k_1$, this gives

$$A_1(r_1, r_2) = \frac{1}{\lambda} \sum_{k_2, m} g(\omega_{k_2 - m}) \overline{g(\omega_{k_2})} \left(f(\omega_{k_2}) \sum_{j_1} \gamma_{j_1} \gamma_{m - j_1} \right) \left(f(\omega_{k_2 + r_2}) \sum_{j_2} \gamma_{j_2} \gamma_{m + r_2 - r_1 - j_2} \right).$$

Next we replace $g(\omega_{k_2 - m})$ with $g(\omega_{k_2})$, and $f(\omega_{k_2 + r_2})$ with $f(\omega_{k_2})$, this gives

$$\begin{aligned} A_1(r_1, r_2) &= \frac{1}{\lambda} \sum_{k_2} g(\omega_{k_2}) \overline{g(\omega_{k_2})} f(\omega_{k_2})^2 \sum_{j_1 + j_2 + j_3 + j_4 = r_2 - r_1} \gamma_{j_1} \gamma_{j_2} \gamma_{j_3} \gamma_{j_4} + O\left(\frac{1 + |r_1| + |r_2|}{\lambda}\right) \\ &= U_\lambda^{(1)}(r_1, r_2) + O\left(\frac{1 + |r_1| + |r_2|}{\lambda}\right), \end{aligned}$$

a similar analysis gives the bound for $A_2(r_1, r_2)$. Applying the same methodology we can also obtain similar bounds for $\lambda \text{cov}[\tilde{Q}_{a,\lambda}(g; r_1), \tilde{Q}_{a,\lambda}(g; r_2)]$.

To prove (ii) we use the same ideas as those used in the proof of Theorem 3.2. By using Lemma A.6 we can show that

$$\lambda \text{cov} \left[\tilde{Q}_{a,\lambda}(g; r_1), \tilde{Q}_{a,\lambda}(g; r_2) \right] = A_1(r_1, r_2) + A_2(r_1, r_2) + O\left(\frac{\lambda}{n}\right)$$

where

$$\begin{aligned} A_1(r_1, r_2) &= \sum_{k_1, k_2 = -a}^a g(\omega_{k_1}) \overline{g(\omega_{k_2})} \text{cov} \left[Z(s_1) \exp(is_1 \omega_{k_1}), Z(s_3) \exp(is_3 \omega_{k_2}) \right] \times \\ &\quad \text{cov} \left[Z(s_2) \exp(-is_2 \omega_{k_1 + r_1}), Z(s_4) \exp(-is_4 \omega_{k_2 + r_2}) \right] \\ A_2(r_1, r_2) &= \sum_{k_1, k_2 = -a}^a g(\omega_{k_1}) \overline{g(\omega_{k_2})} \text{cov} \left[Z(s_1) \exp(is_1 \omega_{k_1}), Z(s_4) \exp(-is_4 \omega_{k_2 + r_2}) \right] \times \\ &\quad \text{cov} \left[Z(s_2) \exp(-is_2 \omega_{k_1 + r_1}), Z(s_3) \exp(is_3 \omega_{k_2}) \right]. \end{aligned}$$

We first analyze $A_1(r_1, r_2)$. Conditioning on s_1, \dots, s_4 gives

$$\begin{aligned} & A_1(r_1, r_2) \\ = & \lambda \sum_{j_1, \dots, j_4 = -\infty}^{\infty} \gamma_{j_1} \gamma_{j_2} \gamma_{j_3} \gamma_{j_4} \sum_{k_1, k_2 = -a}^a g(\omega_{k_1}) \overline{g(\omega_{k_2})} \frac{1}{\lambda^3} \int_{[-\lambda/2, \lambda/2]^4} c(s_1 - s_3) c(s_2 - s_4) \\ & e^{is_1 \omega_{k_1} - is_3 \omega_{k_2}} e^{-is_2 \omega_{k_1} + r_1} e^{is_4 \omega_{k_2} + r_2} e^{-i(s_1 \omega_{j_1} + s_2 \omega_{j_2} + s_3 \omega_{j_3} + s_4 \omega_{j_4})} ds_1 ds_2 ds_3 ds_4. \end{aligned}$$

By using the spectral representation theorem and integrating out s_1, \dots, s_4 we can write the above as

$$\begin{aligned} & A_1(r_1, r_2) \\ = & \frac{\lambda}{(2\pi)^2} \sum_{j_1, \dots, j_4 = -\infty}^{\infty} \gamma_{j_1} \gamma_{j_2} \gamma_{j_3} \gamma_{j_4} \sum_{k_1, k_2 = -a}^a g(\omega_{k_1}) \overline{g(\omega_{k_2})} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) f(y) \operatorname{sinc}\left(\frac{\lambda x}{2} + (k_1 - j_1)\pi\right) \\ & \operatorname{sinc}\left(\frac{\lambda y}{2} - (r_1 + k_1 + j_2)\pi\right) \operatorname{sinc}\left(\frac{\lambda x}{2} + (k_2 + j_3)\pi\right) \operatorname{sinc}\left(\frac{\lambda y}{2} - (r_2 + k_2 - j_4)\pi\right) dx dy \\ = & \frac{1}{\pi^2 \lambda} \sum_{j_1, \dots, j_4 = -\infty}^{\infty} \sum_{k_1, k_2 = -a}^a g(\omega_{k_1}) \overline{g(\omega_{k_2})} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f\left(\frac{2u}{\lambda} - \omega_{k_1 - j_1}\right) f\left(\frac{2v}{\lambda} + \omega_{k_1} + \omega_{r_1 + j_2}\right) \\ & \times \operatorname{sinc}(u) \operatorname{sinc}(u + (k_2 - k_1 + j_1 + j_3)\pi) \operatorname{sinc}(v) \operatorname{sinc}(v + (k_1 - k_2 + r_1 - r_2 + j_2 + j_4)\pi) du dv. \end{aligned}$$

By using the same proof as that given in Lemma 6.1 to approximate the terms inside the sum \sum_{j_1, \dots, j_4} , (and using that $\sum_j |\gamma_j| < \infty$), we can approximate $A_1(r_1, r_2)$ with

$$\begin{aligned} & A_1(r_1, r_2) \\ = & \frac{1}{2\pi^3} \sum_{j_1, \dots, j_4 = -\infty}^{\infty} \gamma_{j_1} \gamma_{j_2} \gamma_{j_4} \gamma_{j_4} \int_{-2\pi a/\lambda}^{2\pi a/\lambda} f(\omega - \omega_{j_1}) f(\omega + \omega_{r_1 + j_2}) \sum_m g(\omega) \overline{g(\omega + \omega_m)} d\omega \\ & \times \int_{\mathbb{R}^2} \operatorname{sinc}(u) \operatorname{sinc}(u + (m + j_1 + j_3)\pi) \operatorname{sinc}(v) \operatorname{sinc}(v + (-m + r_1 - r_2 + j_2 + j_4)\pi) du dv + O(\ell_{\lambda, a, n}). \end{aligned}$$

By orthogonality of the sinc function we see that the above is zero unless $m = -j_1 - j_3$ and $m = r_1 - r_2 + j_2 + j_4$, therefore

$$\begin{aligned} & A_1(r_1, r_2) \\ = & \frac{1}{2\pi} \sum_{j_1 + \dots + j_4 = r_2 - r_1} \gamma_{j_1} \gamma_{j_2} \gamma_{j_4} \gamma_{j_4} \int_{-2\pi a/\lambda}^{2\pi a/\lambda} |g(\omega)|^2 f(\omega - \omega_{j_1}) f(\omega + \omega_{r_1 + j_2}) d\omega + O(\ell_{\lambda, a, n}). \end{aligned}$$

Finally, we replace $f(\omega - \omega_{j_1}) f(\omega + \omega_{r_1 + j_2})$ with $f(\omega)^2$, since $|\gamma_j| = O(|j|^{-2})$, this gives

$$\begin{aligned} & A_1(r_1, r_2) \\ = & \frac{1}{2\pi} \sum_{j_1 + \dots + j_4 = r_2 - r_1} \gamma_{j_1} \gamma_{j_2} \gamma_{j_4} \gamma_{j_4} \int_{-2\pi a/\lambda}^{2\pi a/\lambda} |g(\omega)|^2 f(\omega)^2 d\omega + O(\ell_{\lambda, a, n} + \frac{|r|}{\lambda}), \end{aligned}$$

thus we obtain $U_\lambda^{(1)}(r_1, r_2)$. We can use the same method to obtain an asymptotic expression for $A_2(r_1, r_2)$, $A_4(r_1, r_2)$ and $A_4(r_1, r_2)$. The same methods can be used in the analysis of $d > 1$. This completes the proof. \square

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A Appendix

A.1 Technical Lemmas

We first prove Lemma 5.1, then state four lemmas, which form an important component in the proofs of this paper. Through out this section we use C to denote a finite generic constant.

PROOF of Lemma 5.1 We first prove (19). By using partial fractions and the definition of the sinc function we have

$$\begin{aligned} \int_{-\infty}^{\infty} \text{sinc}(u) \text{sinc}(u+x) du &= \frac{1}{x} \int_{-\infty}^{\infty} \text{sinc}(u) \text{sinc}(u+x) \left(\frac{1}{u} - \frac{1}{u+x} \right) du \\ &= \frac{1}{x} \int_{-\infty}^{\infty} \frac{\text{sinc}(u) \text{sinc}(u+x)}{u} - \int_{-\infty}^{\infty} \frac{\text{sinc}(u) \text{sinc}(u+x)}{u+x} du. \end{aligned}$$

For the second integral we make a change of variables $u' = u+x$, this gives

$$\begin{aligned} \int_{-\infty}^{\infty} \text{sinc}(u) \text{sinc}(u+x) du &= \frac{1}{x} \int_{-\infty}^{\infty} \frac{\text{sinc}(u) \text{sinc}(u+x)}{u} du - \int_{-\infty}^{\infty} \frac{\text{sinc}(u') \text{sinc}(u'-x)}{u'} du' \\ &= \frac{1}{x} \int_{-\infty}^{\infty} \frac{\text{sinc}(u)}{u} [\sin(u+x) - \sin(u-x)] du \\ &= \frac{2 \sin(x)}{x} \int_{-\infty}^{\infty} \frac{\cos(u) \sin(u)}{u} du = \frac{\pi \sin(x)}{x}. \end{aligned}$$

To prove (20), it is clear that for $x = s\pi$ (with $s \neq 0$) $\frac{\pi \sin(s\pi)}{s\pi} = 0$, which gives the result. \square

The following result is used to obtain bounds for the variance and higher order cumulants.

Lemma A.1 Define the function $\ell_p(x) = C/e$ for $|x| \leq e$ and $\ell_p(x) = C \log^p |x|/|x|$ for $|x| \geq e$.

(i) We have

$$\int_{-\infty}^{\infty} \frac{|\sin(x) \sin(x+y)|}{|x(x+y)|} dx \leq \begin{cases} C \frac{\log|y|}{|y|} & |y| \geq e \\ C & |y| < e \end{cases}, \quad (26)$$

$$\int_{-\infty}^{\infty} |\text{sinc}(x)| \ell_p(x+y) dx \leq \ell_{p+1}(y) \quad (27)$$

and

$$\int_{\mathbb{R}^p} \left| \text{sinc} \left(\sum_{j=1}^p x_j \right) \prod_{j=1}^p \text{sinc}(x_j) \right| dx_1 \dots dx_p \leq C, \quad (28)$$

(ii)

$$\sum_{m=-a}^a \int_{-\infty}^{\infty} \left| \frac{\sin^2(x)}{x(x+m\pi)} \right| dx \leq C \log^2 a$$

(iii)

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{m=-a}^a \left| \frac{\sin^2(x)}{|x(x+m\pi)|} \frac{\sin^2(y)}{|y(y+m\pi)|} dx dy \right| \leq C, \quad (29)$$

(iv)

$$\sum_{m_1, \dots, m_{q-1} = -a}^a \int_{\mathbb{R}^q} \left| \prod_{j=1}^{q-1} \text{sinc}(x_j) \text{sinc}(x_j + m_j \pi) \times \right. \\ \left. \text{sinc}(x_q) \text{sinc}(x_q + \pi \sum_{j=1}^{q-1} m_j) \right| \prod_{j=1}^q dx_j \leq C \log^{2(q-2)}(a),$$

where C is a finite generic constant which is independent of a .

PROOF. We first prove (i), equation (26). It is clear that for $|y| \leq e$ that $\int_{-\infty}^{\infty} \frac{|\sin(x) \sin(x+y)|}{|x(x+y)|} dx \leq C$. Therefore we now consider the case $|y| > e$, without loss of generality we prove the result for $y > e$. Partitioning the integral we have

$$\int_{-\infty}^{\infty} \frac{|\sin(x) \sin(x+y)|}{|x(x+y)|} dx = I_1 + I_2 + I_3 + I_4 + I_5,$$

where

$$\begin{aligned} I_1 &= \int_0^y \frac{|\sin(x) \sin(x+y)|}{|x(x+y)|} dx & I_2 &= \int_{-y}^0 \frac{|\sin(x) \sin(x+y)|}{|x(x+y)|} dx \\ I_3 &= \int_{-2y}^{-y} \frac{|\sin(x) \sin(x+y)|}{|x(x+y)|} dx & I_4 &= \int_{-\infty}^{-2y} \frac{|\sin(x) \sin(x+y)|}{|x(x+y)|} dx \\ I_5 &= \int_y^{\infty} \frac{|\sin(x) \sin(x+y)|}{|x(x+y)|} dx \end{aligned}$$

To bound I_1 we note that for $y > 1$ and $x > 0$ that $|\sin(x+y)/(x+y)| \leq 1/y$, thus

$$I_1 = \frac{1}{y} \int_0^y \frac{|\sin(x)|}{|x|} dx \leq C \frac{\log y}{y}.$$

To bound I_2 , we further partition the integral

$$\begin{aligned} I_2 &= \int_{-y}^{-y/2} \frac{|\sin(x) \sin(x+y)|}{|x(x+y)|} dx + \int_{-y/2}^0 \frac{|\sin(x) \sin(x+y)|}{|x(x+y)|} dx \\ &\leq \frac{2}{y} \int_{-y}^{-y/2} \frac{|\sin(x+y)|}{|(x+y)|} dx + \frac{2}{y} \int_{-y/2}^0 \frac{|\sin(x)|}{|x|} dx \leq C \frac{\log y}{y}. \end{aligned}$$

To bound I_3 , we use the bound

$$I_3 \leq \frac{1}{y} \int_{-2y}^{-y} \frac{|\sin(x+y)|}{|(x+y)|} dx \leq C \frac{\log y}{y}.$$

To bound I_4 we use that $\int_y^\infty x^{-2} dx \leq C|y|^{-1}$, thus

$$I_4 \leq \int_{-\infty}^{-2y} \frac{1}{x^2} dx \leq C|y|^{-1}$$

and using a similar argument we have $I_5 \leq C|y|^{-1}$. Altogether, this gives (26).

We now prove (27). It is clear that for $|y| \leq e$ that $\int_{-\infty}^\infty |\text{sinc}(x)| \ell_p(x+y) dx \leq C$. Therefore we now consider the case $|y| > e$, without loss of generality we prove the result for $y > e$. As in (26) we partition the integral

$$\int_{-\infty}^\infty |\text{sinc}(x) \ell_p(x+y)| dx = II_1 + \dots + II_5,$$

where II_1, \dots, II_5 are defined in the same way as I_1, \dots, I_5 just with $|\text{sinc}(x) \ell_p(x+y)|$ replacing $\frac{|\sin(x) \sin(x+y)|}{|x(x+y)|}$. To bound II_1 we note that

$$II_1 = \int_0^y |\text{sinc}(x) \ell_p(x+y)| dx \leq \frac{\log^p(y)}{y} \int_0^y |\text{sinc}(x)| dx \leq C \frac{\log^{p+1}(y)}{y},$$

we use similar method to show $II_2 \leq C \frac{\log^{p+1}(y)}{y}$ and $II_3 \leq C \frac{\log^{p+1}(y)}{y}$. Finally to bound II_4 and II_5 we note that by using a change of variables $x = yz$, we have

$$\begin{aligned} II_5 &= \int_y^\infty \frac{|\sin(x)| \log^p(x+y)}{x(x+y)} dx \leq \int_y^\infty \frac{\log^p(x+y)}{x(x+y)} dx \\ &= \frac{1}{y} \int_1^\infty \left(\frac{[\log(y) + \log(z(z+1))]^p}{z(z+1)} \right) \leq C \frac{\log^p(y)}{y}. \end{aligned}$$

Similarly we can show that $II_4 \leq C \frac{\log^p(y)}{y}$. Altogether, this gives the result.

To prove (28) we recursively apply (27) to give

$$\begin{aligned} \int_{\mathbb{R}^p} |\text{sinc}(x_1 + \dots + x_p)| \prod_{j=1}^p |\text{sinc}(x_j)| dx_1 \dots dx_p &\leq \int_{\mathbb{R}^{p-1}} |\ell_1(x_1 + \dots + x_{p-1})| \prod_{j=1}^{p-1} |\text{sinc}(x_j)| dx_1 \dots dx_{p-1} \\ &\leq \int_{\mathbb{R}} |\ell_{p-1}(x_1) \text{sinc}(x_1)| dx_1 = O(1), \end{aligned}$$

thus we have the required the result.

To bound (ii), without loss of generality we derive a bound over $\sum_{m=1}^a$, the bounds for \sum_{-a}^m is identical. Using (26) we have

$$\begin{aligned} &\sum_{m=1}^a \int_{-\infty}^{\infty} \frac{\sin^2(x)}{|x(x+m\pi)|} dx = \sum_{m=1}^a \int_{-\infty}^{\infty} \frac{|\sin(x) \sin(x+m\pi)|}{|x(x+m\pi)|} dx \\ &\leq \sum_{m=1}^a \ell_1(m\pi) = C \sum_{m=1}^a \frac{\log(m\pi)}{m\pi} \\ &\leq C \log(a\pi) \sum_{m=1}^a \frac{1}{m\pi} = C \log(a\pi) \log(a) \leq C \log^2 a. \end{aligned}$$

Thus we have shown (ii).

To prove (iii) we use (26) to give

$$\begin{aligned} &\sum_{m=-a}^a \left(\int_{-\infty}^{\infty} \frac{\sin^2(x)}{|x(x+m\pi)|} dx \right) \left(\int_{-\infty}^{\infty} \frac{\sin^2(y)}{|y(y+m\pi)|} dy \right) \\ &\leq C \sum_{m=-a}^a \left(\frac{\log m}{m} \right)^2 \leq C \sum_{m=-\infty}^{\infty} \left(\frac{\log m}{m} \right)^2 \leq C. \end{aligned}$$

To prove (iv) we apply (26) to each of the integrals this gives

$$\begin{aligned} &\sum_{m_1, \dots, m_{q-1} = -a}^a \int_{\mathbb{R}^q} \left| \prod_{j=1}^{q-1} \text{sinc}(x_j) \text{sinc}(x_j + m_j \pi) \text{sinc}(x_q) \text{sinc}(x_q + \pi \sum_{j=1}^{q-1} m_j) \right| \prod_{j=1}^q dx_j \\ &\leq \sum_{m_1, \dots, m_{q-1} = -a}^a \ell_1(m_1 \pi) \dots \ell_1(m_{q-1} \pi) \ell_1(m_{q-1} \pi) \ell_1(\pi \sum_{j=1}^{q-1} m_j) \leq C \log^{2(q-2)} a, \end{aligned}$$

thus we obtain the desired result. \square .

The following lemma is used in the proofs of Lemma 3.1, Theorem 3.2 (part (ii)) and Lemma A.5.

Lemma A.2 Suppose h is a function which is absolutely integrable and $|h'(\omega)| \leq \beta_1(\omega)$ (where β_1 is defined in Assumption 2.4(ii)(b)) and $m \in \mathbb{Z}$ and $g(\omega)$ is a bounded function. Then we have

$$\begin{aligned} & \left| \int_{-\infty}^{\infty} \text{sinc}(u) \text{sinc}(u + m\pi) \int_{-b}^b g(\omega) \left(h\left(\omega + \frac{2u}{\lambda}\right) - h(\omega) \right) d\omega du \right| \\ & \leq C\Gamma(b) \frac{\log(\lambda) + \log(m) + \log(\lambda + |m|\pi)}{\lambda}. \end{aligned} \quad (30)$$

where $\Gamma(b) = \int_0^b \beta_1(u) du$, C is a finite constant independent of m and b . If $g(\omega)$ is a bounded function with a bounded first derivative, then we have

$$\begin{aligned} & \left| \int_{-\infty}^{\infty} \text{sinc}(u) \text{sinc}(u + m\pi) \int_{-b}^b h(\omega) \left(g\left(\omega + \frac{2u}{\lambda}\right) - g(\omega) \right) d\omega du \right| \\ & \leq C \frac{\log(\lambda) + \log(m) + \log(\lambda + |m|\pi)}{\lambda}, \end{aligned} \quad (31)$$

where $a \rightarrow \infty$ as $\lambda \rightarrow \infty$.

PROOF. To simplify the notation in the proof, we'll prove (30) for $m > 0$ (the proof for $m \leq 0$ is identical).

The proof is based on considering the cases that $|u| \leq \lambda$ and $|u| > \lambda$ separately. For $|u| \leq \lambda$ we apply the mean value theorem to the difference $h(\omega + \frac{2u}{\lambda}) - h(\omega)$ and for $|u| > \lambda$ we exploit that the integral $\int_{|u| > \lambda} |\text{sinc}(u) \text{sinc}(u + m\pi)| du$ decays as $\lambda \rightarrow \infty$. We now make these argument precise. We start by partitioning the integral

$$\int_{-\infty}^{\infty} \text{sinc}(u) \text{sinc}(u + m\pi) \int_{-b}^b g(\omega) \left(h\left(\omega + \frac{2u}{\lambda}\right) - h(\omega) \right) d\omega du = I_1 + I_2,$$

where

$$\begin{aligned} I_1 &= \int_{|u| > \lambda} \text{sinc}(u) \text{sinc}(u + m\pi) \int_{-b}^b g(\omega) \left(h\left(\omega + \frac{2u}{\lambda}\right) - h(\omega) \right) d\omega du \\ I_2 &= \int_{-\lambda}^{\lambda} \text{sinc}(u) \text{sinc}(u + m\pi) \int_{-b}^b g(\omega) \left(h\left(\omega + \frac{2u}{\lambda}\right) - h(\omega) \right) d\omega du. \end{aligned}$$

We further partition the integral $I_1 = I_{11} + I_{12} + I_{13}$, where

$$\begin{aligned} I_{11} &= \int_{\lambda}^{\infty} \text{sinc}(u) \text{sinc}(u + m\pi) \int_{-b}^b g(\omega) \left(h\left(\omega + \frac{2u}{\lambda}\right) - h(\omega) \right) d\omega du \\ I_{12} &= \int_{-\lambda - m\pi}^{-\lambda} \text{sinc}(u) \text{sinc}(u + m\pi) \int_{-b}^b g(\omega) \left(h\left(\omega + \frac{2u}{\lambda}\right) - h(\omega) \right) d\omega du \\ I_{13} &= \int_{-\lambda}^{-\lambda - m\pi} \text{sinc}(u) \text{sinc}(u + m\pi) \int_{-b}^b g(\omega) \left(h\left(\omega + \frac{2u}{\lambda}\right) - h(\omega) \right) d\omega du \end{aligned}$$

and partition $I_2 = I_{21} + I_{22} + I_{23}$, where

$$\begin{aligned} I_{21} &= \int_{-\lambda}^{\lambda} \text{sinc}(u) \text{sinc}(u + m\pi) \int_{-\min(4|u|/\lambda, b)}^{\min(4|u|/\lambda, b)} g(\omega) \left(h(\omega + \frac{2u}{\lambda}) - h(\omega) \right) d\omega du \\ I_{22} &= \int_{-\lambda}^{\lambda} \text{sinc}(u) \text{sinc}(u + m\pi) \int_{-\min(4|u|/\lambda, b)}^{-b} g(\omega) \left(h(\omega + \frac{2u}{\lambda}) - h(\omega) \right) d\omega du \\ I_{23} &= \int_{-\lambda}^{\lambda} \text{sinc}(u) \text{sinc}(u + m\pi) \int_{\min(4|u|/\lambda, b)}^b g(\omega) \left(h(\omega + \frac{2u}{\lambda}) - h(\omega) \right) d\omega du. \end{aligned}$$

We start by bounding I_1 . Taking absolutes of I_{11} , and using that $h(\omega)$ is absolutely integrable we have

$$I_{11} \leq 2\Gamma \int_{\lambda}^{\infty} \frac{\sin^2(u)}{u(u + m\pi)} du,$$

where $\Gamma = \sup_u |g(u)| \int_{\infty}^{\infty} |h(u)| du$. Since $m > 0$, it is straightforward to show that $\int_{\lambda}^{\infty} \frac{\sin^2(u)}{u(u + m\pi)} du \leq C\lambda^{-1}$, where C is some finite constant. This implies $|I_{11}| \leq 2C\Gamma\lambda^{-1}$. Similarly it can be shown that $|I_{13}| \leq 2C\Gamma\lambda^{-1}$. To bound I_{12} we note that

$$I_{12} \leq \frac{2\Gamma}{\lambda} \int_{-\lambda - m\pi}^{-\lambda} \frac{\sin^2(u)}{(u + m\pi)} = \frac{2\Gamma}{\lambda} \begin{cases} \log(\frac{\lambda}{\lambda - m\pi}) & m\pi < \lambda \\ \log \lambda + \log(m\pi - \lambda) & m\pi > \lambda \end{cases}$$

Thus, we have $I_{12} \leq 2\Gamma\lambda^{-1} [\log \lambda + \log m]$. Altogether, the bounds for I_{11}, I_{12}, I_{13} give

$$|I_1| \leq C \left(\frac{(\log \lambda + \log m)}{\lambda} \right).$$

To bound I_2 we apply the mean value theorem to $h(\omega + \frac{2u}{\lambda}) - h(\omega) \leq \frac{2u}{\lambda} h'(\omega + \gamma(\omega, u) \frac{2u}{\lambda})$, where $0 \leq |\gamma(\omega, u)| \leq 1$. Substituting this into I_{23} gives

$$|I_{23}| \leq \frac{2}{\lambda} \int_{-\lambda}^{\lambda} \frac{\sin^2(u)}{|u + m\pi|} \int_{\min(4|u|/\lambda, b)}^b \left| h'(\omega + \gamma(\omega, u) \frac{2u}{\lambda}) \right| d\omega du.$$

Since the limits of the inner integral are greater than $4u/\lambda$, and the derivative is bounded by $\beta_1(\omega)$, this means $|h'(\omega + \gamma(\omega, u) \frac{2u}{\lambda})| \leq \max[\beta_1(\omega), \beta_1(\omega + \frac{2u}{\lambda})] = \beta_1(\omega)$. Altogether, this gives

$$\begin{aligned} |I_{23}| &\leq \frac{2}{\lambda} \left(\int_{\min(4|u|/\lambda, b)}^b \beta_1(\omega) d\omega \right) \int_{-\lambda}^{\lambda} \frac{\sin^2(u)}{|u + m\pi|} du \\ &\leq \frac{2\Gamma(b) \log(\lambda + m\pi)}{\lambda}, \end{aligned}$$

where $\Gamma(b) = \int_0^b \beta_1(\omega) d\omega$. Using the same method we obtain $I_{22} \leq \frac{2\Gamma(b) \log(\lambda + m\pi)}{\lambda}$. Finally, to bound I_{21} , we cannot bound $h'(\omega + \gamma(\omega, u) \frac{2u}{\lambda})$ by a monotonic function since ω and $\omega + \frac{2u}{\lambda}$

can have different signs. Therefore we simply bound $h'(\omega + \gamma(\omega, u)\frac{2u}{\lambda})$ with a constant, this gives

$$|I_{21}| \leq \frac{8C}{\lambda^2} \int_{-\lambda}^{\lambda} \frac{|u| \sin^2(u)}{(u + m\pi)} du \leq \frac{16C}{\lambda}.$$

Altogether, the bounds for I_{21}, I_{22}, I_{23} give

$$|I_2| \leq C\Gamma(b) \left(\frac{\log \lambda + \log(m) + \log(\lambda + m\pi)}{\lambda} \right).$$

Thus we obtain (30). The proof of (31) is similar, but avoids some of the awkward details that are required to prove (30). \square

Lemma A.3 *Suppose h is a function which is absolutely integrable and $|h'(\omega)| \leq \beta_1(\omega)$ (β_1 is defined in Lemma A.2), $m \in \mathbb{Z}$ and $g(\omega)$ is a bounded function. Furthermore, $m \ll \lambda$ is a fixed constant. Then we have*

$$\begin{aligned} & \int_{\mathbb{R}^3} \text{sinc}(u_1 + u_2 + u_3 + m\pi) \text{sinc}(u_1) \text{sinc}(u_2) \int_{-a/\lambda}^{a/\lambda} g(\omega) \left(h\left(\frac{2u_1}{\lambda} - \omega\right) - h(-\omega) \right) d\omega du_1 du_2 du_3 \\ &= O\left(\frac{\Gamma(a/\lambda) \log^3(\lambda)}{\lambda}\right). \end{aligned}$$

where $\Gamma(b) = \int_0^b \beta_1(u) du$.

PROOF. The proof is very similar to the proof of Lemma A.2. Since $m \ll \lambda$ to simplify notation we prove the result for $m = 0$. We start by partitioning the integral

$$\begin{aligned} & \int_{\mathbb{R}^3} \text{sinc}(u_1 + u_2 + u_3) \text{sinc}(u_1) \text{sinc}(u_2) \text{sinc}(u_3) \\ & \int_{-a/\lambda}^{a/\lambda} g(\omega) \left(h\left(\frac{2u_1}{\lambda} - \omega\right) - h(-\omega) \right) d\omega du_1 du_2 du_3 = I_1 + I_2, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_{|u_1| > \lambda} \int_{\mathbb{R}^2} \text{sinc}(u_1 + u_2 + u_3) \text{sinc}(u_1) \text{sinc}(u_2) \text{sinc}(u_3) \\ & \times \int_{-a/\lambda}^{a/\lambda} g(\omega) \left(h\left(\frac{2u_1}{\lambda} - \omega\right) - h(-\omega) \right) d\omega du_1 du_2 du_3 \\ I_2 &= \int_{|u_1| \leq \lambda} \int_{\mathbb{R}^2} \text{sinc}(u_1 + u_2 + u_3) \text{sinc}(u_1) \text{sinc}(u_2) \text{sinc}(u_3) \\ & \times \int_{-a/\lambda}^{a/\lambda} g(\omega) \left(h\left(\frac{2u_1}{\lambda} - \omega\right) - h(-\omega) \right) d\omega du_1 du_2 du_3. \end{aligned}$$

Taking absolutes and using Lemma A.1, equations (26) and (27) we have

$$|I_1| \leq 4\Gamma \int_{u_1 > \lambda} |\text{sinc}(u_1)| \ell_2(u_1) du_1 \leq C \int_{u > \lambda} \frac{\log^2(u)}{u^2} du,$$

where Γ is defined in Lemma A.2 and C is a finite constant. By making a change of variables $z = \lambda u$, the above becomes

$$|I_1| \leq \frac{4C\Gamma}{\lambda} \int_{u > \lambda} \frac{[\log \lambda + \log(z)]^2}{z^2} dz = O\left(\frac{\log^2(\lambda)}{\lambda}\right).$$

To bound I_2 , just as in Lemma A.1 we decompose it into three parts $I_2 = I_{21} + I_{22} + I_{23}$, where using Lemma A.1, equations (26) and (27) we have the bounds

$$\begin{aligned} |I_{21}| &\leq \int_{|u| \leq \lambda} |\text{sinc}(u)| \ell_2(u) \int_{-\min(a, 4|u|)/\lambda}^{\min(a, 4u)/\lambda} |g(\omega)| \left| h\left(\frac{2u}{\lambda} - \omega\right) - h(-\omega) \right| d\omega du \\ |I_{22}| &\leq \int_{|u| \leq \lambda} |\text{sinc}(u)| \ell_2(u) \int_{-a/\lambda}^{-\min(a, 4|u|)/\lambda} |g(\omega)| \left| h\left(\frac{2u}{\lambda} - \omega\right) - h(-\omega) \right| d\omega du \\ |I_{23}| &\leq \int_{|u| \leq \lambda} |\text{sinc}(u)| \ell_2(u) \int_{a/\lambda}^{\min(a, 4|u|)/\lambda} |g(\omega)| \left| h\left(\frac{2u}{\lambda} - \omega\right) - h(-\omega) \right| d\omega du. \end{aligned}$$

Using the same method used to bound I_{21} in Lemma A.2, we have $|I_{21}| \leq C \log^2(\lambda)/\lambda$. Similarly, the same method to bound I_{22} and I_{23} in Lemma A.2, can be used to show that $I_{22}, I_{23} \leq C \log^3(\lambda)/\lambda$. Having bounded all partitions of the integral, we have the result. \square

A.2 Lemmas required to prove Lemma 6.1 and Theorem 3.2

In this section we give the proofs of the three results used in Lemma 6.1 (which in turn proves Theorem 3.2).

Lemma A.4 *Suppose Assumption 2.4(ii)(a,b) holds. Then we have*

$$|A_1(r_1, r_2) - B_1(r_1 - r_2; r_1)| = O\left(\frac{\log^2(a)}{\lambda}\right)$$

PROOF. To obtain a bound for the difference we use the Lipschitz continuity of f and g to give

$$\begin{aligned}
& |A_1(r_1, r_2) - B_1(r_1 - r_2; r_1)| \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{m=-2a}^{2a} |\text{sinc}(u)\text{sinc}(u + m\pi)\text{sinc}(v)\text{sinc}(v + (m + r_1 - r_2)\pi)| \\
&\quad \underbrace{\left| H_{m,\lambda}\left(\frac{2u}{\lambda}, \frac{2v}{\lambda}; r_1\right) - H_m\left(\frac{2u}{\lambda}, \frac{2v}{\lambda}; r_1\right) \right|}_{\leq C/\lambda} dudv \\
&\leq \frac{C}{\lambda} \sum_{m=-2a}^{2a} \int_{-\infty}^{\infty} |\text{sinc}(u)\text{sinc}(u + m\pi)| du \underbrace{\int_{-\infty}^{\infty} |\text{sinc}(v)\text{sinc}(v + (m + r_1 - r_2)\pi)| dv}_{< \infty} \\
&\leq \frac{C}{\lambda} \sum_{m=-2a}^{2a} \int_{-\infty}^{\infty} |\text{sinc}(u)\text{sinc}(u + m\pi)| du \text{ (from Lemma A.1(ii))} \\
&= O\left(\frac{\log^2 a}{\lambda}\right),
\end{aligned}$$

thus giving the desired result. \square

Lemma A.5 *Suppose Assumption 2.4(ii)(a,b) holds. Then we have*

$$|B_1(s; r) - C_1(s; r)| = O\left(\log^2(a) \left[\frac{\Gamma(a/\lambda)(\log a + \log \lambda)}{\lambda} \right]\right).$$

PROOF. Taking differences, it is easily seen that

$$\begin{aligned}
& B_1(s, r) - C_1(s, r) \\
&= \int_{\mathbb{R}^2} \sum_{m=-2a}^{2a} \text{sinc}(u)\text{sinc}(u + m\pi)\text{sinc}(v)\text{sinc}(v + (m + s)\pi) \\
&\quad \times \frac{1}{(2\pi)} \int_{2\pi \max(-a, -a+m)/\lambda}^{2\pi \min(a, a+m)\lambda} g(\omega) \overline{g(\omega + \omega_m)} \left[f(\omega - \frac{2u}{\lambda}) f(\omega + \frac{2v}{\lambda} + \omega_r) - f(\omega) f(\omega + \omega_r) \right] d\omega dv du \\
&= I_1 + I_2
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^2} \sum_{m=-2a}^{2a} \text{sinc}(u)\text{sinc}(u + m\pi)\text{sinc}(v)\text{sinc}(v + (m + s)\pi) \\
&\quad \times \frac{1}{(2\pi)} \int_{2\pi \max(-a, -a+m)/\lambda}^{2\pi \min(a, a+m)\lambda} g(\omega) \overline{g(\omega + \omega_m)} f(\omega + \frac{2v}{\lambda} + \omega_r) \left[f(\omega - \frac{2u}{\lambda}) - f(\omega) \right] d\omega dv du \\
&= \int_{\mathbb{R}} \sum_{m=-2a}^{2a} \text{sinc}(v)\text{sinc}(v + (m + s)\pi) D_m(v) dv
\end{aligned}$$

and

$$\begin{aligned}
I_2 &= \int_{\mathbb{R}^2} \sum_{m=-2a}^{2a} \text{sinc}(u) \text{sinc}(u + m\pi) \text{sinc}(v) \text{sinc}(v + (m + s)\pi) \\
&\quad \times \frac{1}{(2\pi)} \int_{2\pi \max(-a, -a+m)/\lambda}^{2\pi \min(a, a+m)\lambda} g(\omega) \overline{g(\omega + \omega_m)} f(\omega) \left[f\left(\omega + \frac{2v}{\lambda} + \omega_r\right) - f(\omega + \omega_r) \right] d\omega dv du \\
&= \sum_{m=-2a}^{2a} d_m \int_{\mathbb{R}} \text{sinc}(u) \text{sinc}(u + m\pi) du
\end{aligned}$$

with

$$\begin{aligned}
D_m(v) &= \int_{\mathbb{R}} \text{sinc}(u) \text{sinc}(u + m\pi) \frac{1}{(2\pi)} \int_{2\pi \max(-a, -a+m)/\lambda}^{2\pi \min(a, a+m)\lambda} g(\omega) \overline{g(\omega + \omega_m)} f\left(\omega + \frac{2v}{\lambda} + \omega_r\right) \\
&\quad \times \left[f\left(\omega - \frac{2u}{\lambda}\right) - f(\omega) \right] d\omega du
\end{aligned}$$

and

$$\begin{aligned}
d_m &= \int_{\mathbb{R}} \text{sinc}(v) \text{sinc}(v + (m + s)\pi) \frac{1}{(2\pi)} \int_{2\pi \max(-a, -a+m)/\lambda}^{2\pi \min(a, a+m)\lambda} g(\omega) \overline{g(\omega + \omega_m)} f(\omega) \\
&\quad \times \left[f\left(\omega + \frac{2v}{\lambda} + \omega_r\right) - f(\omega + \omega_r) \right] d\omega dv.
\end{aligned}$$

Since the functions $f(\cdot)$ and $g(\cdot)$ satisfy the conditions stated in Lemma A.2, the lemma can be used to show that

$$\max_{|m| \leq a} \sup_v |D_m(v)| \leq C\Gamma(a/\lambda) \left(\frac{\log a}{a} + \frac{\log \lambda + \log a}{\lambda} \right)$$

and

$$\max_{|m| \leq a} |d_m| \leq C\Gamma(a/\lambda) \left(\frac{\log a}{a} + \frac{\log \lambda + \log a}{\lambda} \right).$$

Substituting these bounds into I_1 and I_2 give

$$\begin{aligned}
|I_1| &\leq C \left(\frac{\Gamma(a/\lambda)(\log \lambda + \log a)}{\lambda} \right) \sum_{m=-2a}^{2a} \int_{-\infty}^{\infty} |\text{sinc}(v) \text{sinc}(v + (m + s)\pi)| dv \\
|I_2| &\leq C \left(\frac{\Gamma(a/\lambda)(\log \lambda + \log a)}{\lambda} \right) \sum_{m=-2a}^{2a} \int_{-\infty}^{\infty} |\text{sinc}(u) \text{sinc}(u + m\pi)| du.
\end{aligned}$$

Therefore, by using Lemma A.1(ii) we have

$$|I_1| \text{ and } |I_2| = O\left(\log^2(a) \left(\frac{\Gamma(a/\lambda)(\log a + \log \lambda)}{\lambda} \right)\right).$$

Since $|B_1(s; r) - C_1(s; r)| \leq |I_1| + |I_2|$ this gives the desired result. \square

A.3 Approximations to the covariance and cumulants of $\tilde{Q}_{a,\lambda}(g; \mathbf{r})$

In this section, our objective is to obtain bounds for $\text{cum}_q(\tilde{Q}_{a,\lambda}(g; \mathbf{r}_1), \dots, \tilde{Q}_{a,\lambda}(g; \mathbf{r}_q))$, these results will be used to prove the asymptotic expression for the variance of $\tilde{Q}_{a,\lambda}(g; \mathbf{r})$ (given in Section 6) and asymptotic normality of $\tilde{Q}_{a,\lambda}(g; \mathbf{r})$. Fox and Taqqu (1987), Dahlhaus (1989), Giraitis and Surgailis (1990) (see also Taqqu and Peccati (2011)) have developed techniques for dealing with the cumulants of sums of periodograms of Gaussian (discrete time) time series, and one would have expected that these results could be used here. However, in our setting there are a few differences that we now describe (i) despite the spatial random being Gaussian the locations are randomly sampled, thus the composite process $Z(\mathbf{s})$ is not Gaussian (we can only exploit the Gaussianity when we condition on the location) (ii) the random field is defined over \mathbb{R}^d (not \mathbb{Z}^d) (iii) the number of terms in the sums $\tilde{Q}_{a,\lambda}(\cdot)$ is not necessarily the sample size. Unfortunately, these differences make it difficult to apply the above mentioned results to our setting.

As a simple motivation we first consider $\text{var}[\tilde{Q}_{a,\lambda}(1, 0)]$ in the case $d = 1$, noting that the methods discussed below generalize to higher order cumulants. By using indecomposable partitions we have

$$\begin{aligned}
& \text{var}[\tilde{Q}_{a,\lambda}(1, 0)] \\
&= \frac{1}{n^4} \sum_{j_1, j_2, j_3, j_4=1}^n \sum_{k_1, k_2=-a}^a \text{cov}[Z(s_{j_1})Z(s_{j_2})\exp(i\omega_{k_1}(s_{j_1} - s_{j_2})), Z(s_{j_3})Z(s_{j_4})\exp(i\omega_{k_2}(s_{j_3} - s_{j_4}))] \\
&= \frac{1}{n^4} \sum_{j_1, j_2, j_3, j_4=1}^n \sum_{k_1, k_2=-a}^a \left(\text{cov}[Z(s_{j_1})e^{is_{j_1}\omega_{k_1}}, Z(s_{j_3})e^{is_{j_3}\omega_{k_2}}] \text{cov}[Z(s_{j_2})e^{-is_{j_2}\omega_{k_1}}, Z(s_{j_4})e^{-is_{j_4}\omega_{k_2}}] \right. \\
&\quad + \text{cov}[Z(s_{j_1})e^{is_{j_1}\omega_{k_1}}, Z(s_{j_4})e^{-is_{j_4}\omega_{k_2}}] \text{cov}[Z(s_{j_2})e^{-is_{j_2}\omega_{k_1}}, Z(s_{j_3})e^{is_{j_3}\omega_{k_2}}] \\
&\quad \left. + \text{cum}[Z(s_{j_1})e^{is_{j_1}\omega_{k_1}}, Z(s_{j_2})e^{-is_{j_2}\omega_{k_1}}, Z(s_{j_3})e^{is_{j_3}\omega_{k_2}}, Z(s_{j_4})e^{-is_{j_4}\omega_{k_2}}] \right). \tag{32}
\end{aligned}$$

In order to evaluate the covariances in the above we condition on the locations $\{s_j\}$. To evaluate the fourth order cumulant of the above we appeal to a generalisation of the conditional variance method. This expansion was first derived in Brillinger (1969), and in the general setting it is stated as

$$\text{cum}(Y_1, Y_2, \dots, Y_q) = \sum_{\pi} \text{cum}[\text{cum}(Y_{\pi_1}|s_1, \dots, s_q), \dots, \text{cum}(Y_{\pi_b}|s_1, \dots, s_q)], \tag{33}$$

where the sum is over all partitions π of $\{1, \dots, q\}$ and $\{\pi_1, \dots, \pi_b\}$ are all the blocks in the partition π . We use (33) to evaluate $\text{cum}[Z(s_{j_1})e^{is_{j_1}\omega_{k_1}}, \dots, Z(s_{j_q})e^{is_{j_q}\omega_{k_q}}]$, where $Y_i = Z(s_{j_i})e^{is_{j_i}\omega_{k_i}}$ and we condition on the locations $\{s_j\}$. Using this decomposition we can see

that because the spatial process is Gaussian, $\text{cum}[Z(s_{j_1})e^{is_{j_1}\omega_{k_1}}, \dots, Z(s_{j_q})e^{is_{j_q}\omega_{k_q}}]$ can only be composed of cumulants of covariances conditioned on the locations. Moreover, if s_1, \dots, s_q are independent then by using the same reasoning we see that $\text{cum}[Z(s_1)e^{is_1\omega_{k_1}}, \dots, Z(s_q)e^{is_q\omega_{k_q}}] = 0$. Therefore, $\text{cum}[Z(s_{j_1})e^{is_{j_1}\omega_{k_1}}, Z(s_{j_2})e^{-is_{j_2}\omega_{k_1}}, Z(s_{j_3})e^{is_{j_3}\omega_{k_2}}, Z(s_{j_4})e^{-is_{j_4}(\omega_{k_2})}]$ will only be non-zero if some elements of $s_{j_1}, s_{j_2}, s_{j_3}, s_{j_4}$ are dependent. Using these rules we have

$$\begin{aligned}
& \text{var}[\tilde{Q}_{a,\lambda}(1,0)] \\
&= \frac{1}{n^4} \sum_{j_1, j_2, j_3, j_4 \in \mathcal{D}_4} \sum_{k_1, k_2 = -a}^a \left(\text{cov}[Z(s_{j_1})e^{is_{j_1}\omega_{k_1}}, Z(s_{j_3})e^{is_{j_3}\omega_{k_2}}] \text{cov}[Z(s_{j_2})e^{-is_{j_2}\omega_{k_1}}, Z(s_{j_4})e^{-is_{j_4}\omega_{k_2}}] \right. \\
&\quad \left. + \text{cov}[Z(s_{j_1})e^{is_{j_1}\omega_{k_1}}, Z(s_{j_4})e^{-is_{j_4}\omega_{k_2}}] \text{cov}[Z(s_{j_2})e^{-is_{j_2}\omega_{k_1}}, Z(s_{j_3})e^{is_{j_3}\omega_{k_2}}] \right) \\
&\quad + \frac{1}{n^4} \sum_{j_1, j_2, j_3, j_4 \in \mathcal{D}_3} \sum_{k_1, k_2 = -a}^a \left(\text{cov}[Z(s_{j_1})e^{is_{j_1}\omega_{k_1}}, Z(s_{j_3})e^{is_{j_3}\omega_{k_2}}] \text{cov}[Z(s_{j_2})e^{-is_{j_2}\omega_{k_1}}, Z(s_{j_4})e^{-is_{j_4}\omega_{k_2}}] \right. \\
&\quad \left. + \text{cov}[Z(s_{j_1})e^{is_{j_1}\omega_{k_1}}, Z(s_{j_4})e^{-is_{j_4}\omega_{k_2}}] \text{cov}[Z(s_{j_2})e^{-is_{j_2}\omega_{k_1}}, Z(s_{j_3})e^{is_{j_3}\omega_{k_2}}] \right) \\
&\quad + \frac{1}{n^4} \sum_{j_1, j_2, j_3, j_4 \in \mathcal{D}_3} \sum_{k_1, k_2 = -a}^a \text{cum}[Z(s_{j_1})e^{is_{j_1}\omega_{k_1}}, Z(s_{j_2})e^{-is_{j_2}\omega_{k_1}}, Z(s_{j_3})e^{is_{j_3}\omega_{k_2}}, Z(s_{j_4})e^{-is_{j_4}(\omega_{k_2})}], \quad (34)
\end{aligned}$$

where $\mathcal{D}_4 = \{j_1, \dots, j_4 = \text{all } j\text{'s are different}\}$, $\mathcal{D}_3 = \{j_1, \dots, j_4; \text{two } j\text{'s are the same but } j_1 \neq j_2 \text{ and } j_3 \neq j_4\}$ (noting that by definition of $\tilde{Q}_{a,\lambda}(1,0)$ more than two elements in $\{j_1, \dots, j_4\}$ cannot be the same). We observe that $|\mathcal{D}_4| = O(n^4)$ and $|\mathcal{D}_3| = O(n^3)$, where $|\cdot|$ denotes the cardinality of a set. We will show that the second and third terms are asymptotically negligible with respect to the first term. To show this we require the following lemma.

Lemma A.6 *Suppose Assumptions 2.4 and 2.1 hold. Then we have*

$$\sum_{k_1, k_2 = -n}^n \text{cum}[Z(s_1)e^{is_1\omega_{k_1}}, Z(s_2)e^{-is_2\omega_{k_1}}, Z(s_3)e^{-is_3\omega_{k_2}}, Z(s_1)e^{is_1\omega_{k_2}}] = O(1) \quad (35)$$

$$\sum_{k_1, k_2} \text{cov}[Z(s_1)e^{is_1\omega_{k_1}}, Z(s_1)e^{-is_1\omega_{k_2}+r_2}] \text{cov}[Z(s_2)e^{-is_2\omega_{k_1}+r_1}, Z(s_3)e^{is_3\omega_{k_2}}] = O(1). \quad (36)$$

PROOF. To show (35) we use conditioning cumulants (see (33)). By using the conditional

cumulant expansion and Gaussianity of $Z(s)$ conditioned on the location we have

$$\begin{aligned}
& \sum_{k_1, k_2 = -a}^a \text{cum}[Z(s_1)e^{is_1\omega_{k_1}}, Z(s_2)e^{-is_2\omega_{k_1}}, Z(s_3)e^{-is_3\omega_{k_2}}, Z(s_1)e^{is_1\omega_{k_2}}] \\
&= \sum_{k_1, k_2 = -a}^a \text{cum}[c(s_1 - s_2)e^{is_1\omega_{k_1} - is_2\omega_{k_1}}, c(s_3 - s_1)e^{-is_3\omega_{k_2} + is_1\omega_{k_2}}] + \\
& \quad \sum_{k_1, k_2 = -a}^a \text{cum}[c(s_1 - s_3)e^{is_1\omega_{k_1} - is_3\omega_{k_2}}, c(s_2 - s_1)e^{-is_2\omega_{k_1} + is_1\omega_{k_2}}] + \\
& \quad \underbrace{\sum_{k_1, k_2 = -a}^a \text{cum}[c(0)e^{is_1(\omega_{k_1} + \omega_{k_2})}, c(s_2 - s_3)e^{-is_2\omega_{k_1} - is_3\omega_{k_2}}]}_{=0 \text{ (since } s_1 \text{ is independent of } s_2 \text{ and } s_3)} \\
&= I_1 + I_2.
\end{aligned}$$

Writing I_1 in terms of expectations and using the spectral representation of the covariance we have

$$\begin{aligned}
I_1 &= \sum_{k_1, k_2 = -a}^a \left(\mathbb{E}[c(s_1 - s_3)c(s_1 - s_2)e^{is_1(\omega_{k_1} + \omega_{k_2})}e^{-is_3\omega_{k_2}}e^{-is_2\omega_{k_1}}] - \mathbb{E}[c(s_1 - s_3)e^{is_1\omega_{k_1} - is_3\omega_{k_2}}] \right. \\
& \quad \left. \times \mathbb{E}[c(s_1 - s_2)e^{-is_2\omega_{k_1} + is_1\omega_{k_2}}] \right) \\
&= \sum_{k_1, k_2 = -a}^a \int \int f(x)f(y) \text{sinc}\left(\frac{\lambda}{2}(x+y) + (k_1 + k_2)\pi\right) \text{sinc}\left(\frac{\lambda}{2}x + k_1\pi\right) \text{sinc}\left(\frac{\lambda}{2}y + k_2\pi\right) dx dy - \\
& \quad \sum_{k_1, k_2 = -a}^a \int \int f(x)f(y) \text{sinc}\left(\frac{\lambda}{2}x + k_1\pi\right) \text{sinc}\left(\frac{\lambda}{2}x + k_2\pi\right) \text{sinc}\left(\frac{\lambda}{2}y + k_1\pi\right) \text{sinc}\left(\frac{\lambda}{2}y + k_2\pi\right) dx dy \\
&= E_1 - E_2.
\end{aligned}$$

To bound E_1 we make a change of variables $u = \frac{\lambda x}{2} + k_1\pi$, $v = \frac{\lambda y}{2} + k_2\pi$, and replace sum with integral to give

$$\begin{aligned}
E_1 &= \frac{4}{\lambda^2} \int \int \sum_{k_1, k_2 = -a}^a f\left(\frac{2u}{\lambda} - \omega_{k_1}\right) f\left(\frac{2v}{\lambda} - \omega_{k_2}\right) \text{sinc}(u+v) \text{sinc}(u) \text{sinc}(v) du dv \\
&= 4 \int \int \text{sinc}(u+v) \text{sinc}(u) \text{sinc}(v) \left(\int_{-a/\lambda}^{a/\lambda} \int_{-a/\lambda}^{a/\lambda} f\left(\frac{2u}{\lambda} - \omega_1\right) f\left(\frac{2v}{\lambda} - \omega_2\right) d\omega_1 d\omega_2 \right) du dv + O\left(\frac{1}{\lambda}\right).
\end{aligned}$$

Let $G(\frac{2u}{\lambda}) = \int_{-a/\lambda}^{a/\lambda} f(\frac{2u}{\lambda} - \omega) d\omega$, then substituting this into the above and using equation (28) in Lemma A.1 we have

$$E_1 = 4 \int \int \text{sinc}(u+v) \text{sinc}(u) \text{sinc}(v) G\left(\frac{2u}{\lambda}\right) G\left(\frac{2v}{\lambda}\right) du dv + O\left(\frac{1}{\lambda}\right).$$

To bound E_2 we use a similar technique and Lemma A.1(iii) to give $E_2 = O(\frac{1}{\lambda})$. Altogether, this gives $I_1 = O(1)$. The same proof can be used to show that $I_2 = O(1)$. Altogether this gives (35).

To bound (36), we observe that if $k_1 \neq k_2$, then $\text{cov}[Z(s_1)e^{is_1\omega_{k_1}}, Z(s_1)e^{is_1\omega_{k_2}}] = E(c(0)e^{is(\omega_{k_1}-\omega_{k_2})}) = 0$. Therefore (36) can be reduced to

$$\begin{aligned}
& \sum_{k_1, k_2 = -a}^a \text{cov}[Z(s_1)e^{is_1\omega_{k_1}}, Z(s_1)e^{is_1\omega_{k_2}}] \text{cov}[Z(s_2)e^{-is_2\omega_{k_1}}, Z(s_3)e^{is_3\omega_{k_2}}] \\
&= \frac{c(0)}{2\pi} \sum_{k=-a}^a \text{cov}[Z(s_2)e^{-is_2\omega_{k_1}}, Z(s_3)e^{is_3\omega_{k_2}}] \\
&= \frac{c(0)}{2\pi} \int_{-\infty}^{\infty} \sum_{k=-a}^a f(x) \text{sinc}(\frac{\lambda x}{2} + k\pi) \text{sinc}(\frac{\lambda x}{2} + k\pi) dx \quad (\text{let } \frac{\lambda x}{2} + k\pi) \\
&= c(0) \int_{-\infty}^{\infty} \frac{1}{\pi\lambda} \sum_{k=-a}^a f(\frac{2\omega}{\lambda} - \omega_k) \text{sinc}^2(\omega) d\omega \\
&= \frac{c(0)}{\pi} \int_{-\infty}^{\infty} \text{sinc}^2(\omega) \left(\int_{-a/\lambda}^{a/\lambda} f(\frac{\lambda\omega}{2} - x) dx \right) d\omega + O(\frac{1}{\lambda}) = O(1),
\end{aligned}$$

thus proving (36). \square

We now derive an expression for $\text{var}[\tilde{Q}_{a,\lambda}(1, 0)]$, by using Lemma A.6 we have

$$\begin{aligned}
& \text{var}[\tilde{Q}_{a,\lambda}(1, 0)] \\
&= \frac{1}{n^4} \sum_{j_1, j_2, j_3, j_4 \in \mathcal{D}_4} \sum_{k_1, k_2 = -a}^a \left(\text{cov}[Z(s_{j_1})e^{is_{j_1}\omega_{k_1}}, Z(s_{j_3})e^{is_{j_3}\omega_{k_2}}] \text{cov}[Z(s_{j_2})e^{-is_{j_2}\omega_{k_1}}, Z(s_{j_4})e^{-is_{j_4}\omega_{k_2}}] \right. \\
& \quad \left. + \text{cov}[Z(s_{j_1})e^{is_{j_1}\omega_{k_1}}, Z(s_{j_4})e^{-is_{j_4}\omega_{k_2}}] \text{cov}[Z(s_{j_2})e^{-is_{j_2}\omega_{k_1}}, Z(s_{j_3})e^{is_{j_3}\omega_{k_2}}] \right) + O(\frac{1}{n}). \tag{37}
\end{aligned}$$

In Lemma 3.1 we have shown that the covariance terms above are of order $O(\lambda^{-1})$, thus dominating the forth order cumulant terms which is of order $O(n^{-1})$ (so long as $\lambda \ll n$).

Lemma A.7 *Suppose that $\{Z(\mathbf{s}); \mathbf{s} \in \mathbb{R}^d\}$ is a Gaussian random process and $\{\mathbf{s}_j\}$ are iid random variables. Then the following hold:*

- (i) *As we mentioned above by using the that spatial process is Gaussian and (33), $\text{cum}[Z(\mathbf{s}_{j_1})e^{is_{j_1}\omega_{\mathbf{k}_1}}, \dots, Z(\mathbf{s}_{j_{2q+1}})e^{is_{j_{2q+1}}\omega_{\mathbf{k}_{2q+1}}}]$ can be written as the sum of products of cumulants of the spatial covariance conditioned on location. Therefore, it is easily seen that the odd order cumulant $\text{cum}_{2q+1}[Z(\mathbf{s}_{j_1}) \exp(is_{j_1}\omega_{\mathbf{k}_1}), \dots, Z(\mathbf{s}_{j_{2q+1}}) \exp(is_{j_{2q+1}}\omega_{\mathbf{k}_{2q+1}})] = 0$ for all q and regardless of $\{\mathbf{s}_j\}$ being dependent or not.*

(ii) By the same reasoning given above, $\text{cum}[Z(\mathbf{s}_{j_1})e^{i\mathbf{s}_{j_1}\boldsymbol{\omega}_{\mathbf{k}_1}}, \dots, Z(\mathbf{s}_{j_{2q}})e^{i\mathbf{s}_{j_{2q}}\boldsymbol{\omega}_{\mathbf{k}_{2q}}}]$, we observe that if more than $(q+1)$ locations $\{s_j; j=1, \dots, 2q\}$ are independent, then $\text{cum}_{2q}[Z(\mathbf{s}_{j_1})\exp(i\mathbf{s}_{j_1}\boldsymbol{\omega}_{\mathbf{k}_1}), \dots, Z(\mathbf{s}_{j_{2q}})\exp(i\mathbf{s}_{j_{2q}}\boldsymbol{\omega}_{\mathbf{k}_{2q}}))] = 0$.

Lemma A.8 Suppose Assumptions 2.1, 2.2 and 2.4(i)(a,b) or (ii)(a,b) are satisfied, and $d = 1$. Then we have

$$\text{cum}_3[\tilde{Q}_{a,\lambda}(g, r)] = O\left(\frac{\log^2(a)}{\lambda^2}\right) \quad (38)$$

with $\frac{\lambda}{n \log^2(a)} \rightarrow 0$, $\log^2(a)/\lambda \rightarrow \infty$ as $\lambda \rightarrow \infty$, $n \rightarrow \infty$ and $a \rightarrow \infty$.

PROOF. We prove the result for $\text{cum}_3(\tilde{Q}_{a,\lambda}(1, 0))$, noting that under Assumption 2.4 the proof is identical for general g and r . We first expand $\text{cum}_3[\tilde{Q}_{a,\lambda}(1, 0)]$ using indecomposable partitions. Using that Lemma A.7(i) we note that the third order cumulant is zero, therefore

$$\begin{aligned} & \text{cum}_3[\tilde{Q}_{a,\lambda}(1, 0)] \\ &= \frac{1}{n^6} \sum_{\underline{j} \in \mathcal{D}} \sum_{k_1, k_2, k_3 = -a}^a \text{cum}[Z(s_{j_1})Z(s_{j_2})\exp(i\omega_{k_1}(s_{j_1} - s_{j_2})), \\ & \quad Z(s_{j_3})Z(s_{j_4})\exp(i\omega_{k_2}(s_{j_3} - s_{j_4})), Z(s_{j_5})Z(s_{j_6})\exp(i\omega_{k_3}(s_{j_5} - s_{j_6}))] \\ &= \frac{1}{n^6} \sum_{\underline{j} \in \mathcal{D}} \sum_{\pi_1 \in \mathcal{P}_1} A_{2,2,2}^j(\pi_1) + \frac{1}{n^6} \sum_{\underline{j} \in \mathcal{D}} \sum_{\pi_2 \in \mathcal{P}_2} A_{4,2}^j(\pi_2) + \frac{1}{n^6} \sum_{\underline{j} \in \mathcal{D}} A_6^j \\ &= B_{2,2,2} + B_{4,2} + B_6, \end{aligned}$$

where $\mathcal{D} = \{j_1, \dots, j_6 \in \{1, \dots, n\} \text{ but } j_1 \neq j_2, j_3 \neq j_4, j_5 \neq j_6\}$, $A_{2,2,2}^j$ (by definition of $\tilde{Q}_{a,\lambda}$) consists of only covariances and \mathcal{P}_1 is the set of all covariance pairs of indecomposable partitions of $\{(1, 2), (3, 4), (5, 6)\}$, $A_{4,2}^j$ consists of only covariance and 4th order cumulant products and \mathcal{P}_2 is the set of all 4th order and 2nd order cumulant indecomposable partitions of $\{(1, 2), (3, 4), (5, 6)\}$ and A_6^j is the 6th order cumulant. Examples of A 's are given below

$$\begin{aligned} A_{2,2,2}^j(\pi_{1,1}) &= \sum_{k_1, k_2, k_3 = -a}^a \text{cov}(Z(s_{j_1})e^{i\mathbf{s}_{j_1}\boldsymbol{\omega}_{\mathbf{k}_1}}, Z(s_{j_3})e^{i\mathbf{s}_{j_3}\boldsymbol{\omega}_{\mathbf{k}_2}}) \text{cov}(Z(s_{j_2})e^{-i\mathbf{s}_{j_2}\boldsymbol{\omega}_{\mathbf{k}_1}}, Z(s_{j_5})e^{i\mathbf{s}_{j_5}\boldsymbol{\omega}_{\mathbf{k}_3}}) \\ & \quad \times \text{cov}(Z(s_{j_4})e^{-i\mathbf{s}_{j_4}\boldsymbol{\omega}_{\mathbf{k}_2}}, Z(s_{j_6})e^{i\mathbf{s}_{j_6}\boldsymbol{\omega}_{\mathbf{k}_3}}) \\ &= \sum_{k_1, k_2, k_3 = -a}^a \mathbb{E}[c(s_{j_1} - s_{j_3})e^{i(s_{j_1}\boldsymbol{\omega}_{\mathbf{k}_1} + s_{j_3}\boldsymbol{\omega}_{\mathbf{k}_2})}] \mathbb{E}[c(s_{j_2} - s_{j_5})e^{i(-s_{j_2}\boldsymbol{\omega}_{\mathbf{k}_1} + s_{j_5}\boldsymbol{\omega}_{\mathbf{k}_3})}] \times \\ & \quad \mathbb{E}[c(s_{j_4} - s_{j_6})e^{-i(s_{j_4}\boldsymbol{\omega}_{\mathbf{k}_2} + s_{j_6}\boldsymbol{\omega}_{\mathbf{k}_3})}], \end{aligned} \quad (39)$$

$$\begin{aligned}
A_{4,2}^j(\pi_{2,1}) &= \sum_{k_1, k_2, k_3 = -a}^a \text{cum}[Z(s_{j_2}) \exp(-is_{j_2}\omega_{k_1}), Z(s_{j_6}) \exp(-is_{j_6}\omega_3)] \\
&\quad \text{cum}[Z(s_{j_1}) \exp(is_{j_1}\omega_{k_1}), Z(s_{j_3}) \exp(-is_{j_3}\omega_{k_2}), Z(s_{j_4}) \exp(-is_{j_4}\omega_{k_2}), Z(s_{j_5}) \exp(is_{j_5}\omega_{k_3})],
\end{aligned} \tag{40}$$

$$\begin{aligned}
A_6^j &= \sum_{k_1, k_2, k_3 = -a}^a \text{cum}[Z(s_{j_1}) \exp(is_{j_1}\omega_{k_1}), Z(s_{j_2}) \exp(-is_{j_2}\omega_{k_1}), Z(s_{j_3}) \exp(is_{j_3}\omega_{k_2}), \\
&\quad Z(s_{j_4}) \exp(-is_{j_4}\omega_{k_2}), Z(s_{j_5}) \exp(is_{j_5}\omega_{k_3}), Z(s_{j_6}) \exp(-is_{j_6}\omega_3)],
\end{aligned} \tag{41}$$

where $\underline{j} = (j_1, \dots, j_6)$.

Bound for B_{222}

We will show that B_{222} is the leading term in $\text{cum}_3(\tilde{Q}_{a,\lambda}(g; 0))$. The set \mathcal{D} is split into four sets, \mathcal{D}_6 where all the elements of \underline{j} are different, and for $3 \leq i \leq 5$, \mathcal{D}_i where i elements in \underline{j} are the same, such that

$$B_{2,2,2} = \frac{1}{n^6} \sum_{i=0}^3 \sum_{\underline{j} \in \mathcal{D}_{6-i}} \sum_{\pi_1 \in \mathcal{P}_1} A_{2,2,2}^j(\pi_1).$$

We start by bounding the partition given in (39), we later explain how the same bounds can be obtained for other indecomposable partitions in \mathcal{P}_1 . By using the spectral representation of the covariance and that $|\mathcal{D}_6| = O(n^6)$, it is straightforward to show that

$$\begin{aligned}
&\frac{1}{n^6} \sum_{\underline{j} \in \mathcal{D}_6} A_{2,2,2}^j(\pi_{1,1}) \\
&= \sum_{k_1, k_2, k_3} \frac{C_6}{(2\pi)^3 \lambda^6} \int_{\mathbb{R}^3} \int_{[-\lambda/2, \lambda/2]^6} f(x) f(y) f(z) \exp(i\omega_{k_1}(s_1 - s_2)) \times \\
&\quad \exp(i\omega_{k_2}(s_3 - s_4)) \times \exp(i\omega_{k_3}(s_5 - s_6)) \\
&\quad \exp(ix(s_1 - s_3)) \exp(iy(s_4 - s_6)) \exp(iz(s_2 - s_5)) \prod_{j=1}^3 ds_{2j-1} ds_{2j} dx dy dz \\
&= \frac{C_6}{(2\pi)^3} \sum_{k_1, k_2, k_3} \int_{\mathbb{R}^3} f(x) f(y) f(z) \text{sinc}\left(\frac{\lambda x}{2} + k_1 \pi\right) \text{sinc}\left(\frac{\lambda z}{2} - k_1 \pi\right) \times \\
&\quad \text{sinc}\left(\frac{\lambda y}{2} - k_2 \pi\right) \text{sinc}\left(\frac{\lambda y}{2} - k_2 \pi\right) \text{sinc}\left(\frac{\lambda z}{2} - k_3 \pi\right) \text{sinc}\left(\frac{\lambda y}{2} + k_3 \pi\right) dx dy dz,
\end{aligned} \tag{42}$$

where $C_6 = n(n-1)\dots(n-5)/n^6$. By changing variables $x = \frac{\lambda x}{2} + k_1\pi$, $y = \frac{\lambda y}{2} - k_2\pi$ and $z = \frac{\lambda z}{2} - k_3\pi$ we have

$$\begin{aligned} & \frac{1}{n^6} \sum_{\underline{j} \in \mathcal{D}_6} A_{2,2,2}^j(\pi_{1,1}) \\ &= \frac{C_6}{(2\pi)^3} \sum_{k_1, k_2, k_3} \frac{1}{\lambda^3} \int_{\mathbb{R}^3} f\left(\frac{2x}{\lambda} - \omega_{k_1}\right) f\left(\frac{2y}{\lambda} + \omega_{k_2}\right) f\left(\frac{2z}{\lambda} + \omega_{k_3}\right) \text{sinc}(x) \text{sinc}(z) \text{sinc}(y) \\ & \quad \text{sinc}(x - (k_2 + k_1)\pi) \text{sinc}(z - (k_1 - k_3)\pi) \text{sinc}(y + (k_3 + k_2)\pi) dx dy dz. \end{aligned} \quad (43)$$

In order to understand how this case can generalise to other partitions in \mathcal{P}_1 , we represent the k s inside the sinc function using the the linear equations

$$\begin{pmatrix} -1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}, \quad (44)$$

where we observe that the above is a rank two matrix. Based on this we make the following change of variables $k_1 = k_1$, $m_1 = k_2 + k_1$ and $m_2 = k_1 - k_3$, and rewrite the sum as

$$\begin{aligned} & \frac{1}{n^6} \sum_{\underline{j} \in \mathcal{D}_6} A_{2,2,2}^j(\pi_{1,1}) \\ &= \frac{C_6}{(2\pi)^3} \sum_{k_1, m_1, m_2} \frac{1}{\lambda^3} \int_{\mathbb{R}^3} f\left(\frac{2x}{\lambda} - \omega_{k_1}\right) f\left(\frac{2y}{\lambda} + \omega_{m_1 - k_1}\right) f\left(\frac{2z}{\lambda} - \omega_{k_1 - m_2}\right) \text{sinc}(x) \text{sinc}(z) \text{sinc}(y) \\ & \quad \text{sinc}(x - m_1\pi) \text{sinc}(y - m_2\pi) \text{sinc}(z + (m_1 - m_2)\pi) dx dy dz \\ &= \frac{C_6}{\lambda^2} \sum_{m_1, m_2} \int_{\mathbb{R}^3} \text{sinc}(x) \text{sinc}(x - m_1\pi) \text{sinc}(y) \text{sinc}(y + (m_1 - m_2)\pi) \\ & \quad \text{sinc}(z) \text{sinc}(z - m_2\pi) \frac{1}{\lambda} \sum_{k_1} f\left(\frac{\lambda x}{2} - \omega_{k_1}\right) f\left(\frac{\lambda y}{2} + \omega_{m_1 - k_1}\right) f\left(\frac{\lambda z}{2} - \omega_{k_1 - m_1}\right) dx dy dz. \end{aligned} \quad (45)$$

Finally, we apply Lemma A.1(iv) to give

$$\frac{1}{n^6} \sum_{\underline{j} \in \mathcal{D}_6} A_{2,2,2}^j(\pi_{1,1}) = O\left(\frac{\log^2(a)}{\lambda^2}\right). \quad (46)$$

The above only gives the bound for one partition of \mathcal{P}_1 , but we now show that the same bound applies to all the other partitions. Looking back at (43) and comparing with (45), the reason that only one of the three λ s in the denominator of (43) gets ‘swallowed’ is because there are two independent m s in the sinc function of (43) (which allow us to apply Lemma A.1(iv)). Furthermore, there are two independent m s in the sinc function is because the matrix in (44) has rank two. However, it can be shown that all indecomposable partitions

of \mathcal{P}_1 correspond to rank two matrices (for a proof see equation (A.13) in Deo and Chen (2000)). Thus all indecomposable partitions in \mathcal{P}_1 will have the same order, which altogether gives

$$\frac{1}{n^6} \sum_{\underline{j} \in \mathcal{D}_6} \sum_{\pi_1 \in \mathcal{P}_1} A_{2,2,2}^j(\pi_1) = O\left(\frac{\log^2(a)}{\lambda^2}\right).$$

Now we consider the case that $\underline{j} \in \mathcal{D}_5$. In this case, there are two ‘typical’ cases

$$\begin{aligned} & \text{cov}(Z(s_{j_1})e^{is_{j_1}\omega_{k_1}}, Z(s_{j_3})e^{is_{j_3}\omega_{k_2}}) \text{cov}(Z(s_{j_1})e^{-is_{j_1}\omega_{k_1}}, Z(s_{j_5})e^{is_{j_5}\omega_{k_3}}) \text{cov}(Z(s_{j_4})e^{-is_{j_4}\omega_{k_2}}, Z(s_{j_6})e^{is_{j_6}\omega_{k_3}}) \\ & \text{and} \\ & \text{cov}(Z(s_{j_1})e^{is_{j_1}\omega_{k_1}}, Z(s_{j_1})e^{is_{j_1}\omega_{k_2}}) \text{cov}(Z(s_{j_2})e^{-is_{j_2}\omega_{k_1}}, Z(s_{j_5})e^{is_{j_5}\omega_{k_3}}) \text{cov}(Z(s_{j_4})e^{-is_{j_4}\omega_{k_2}}, Z(s_{j_6})e^{is_{j_6}\omega_{k_3}}). \end{aligned}$$

The first leads to the same bound as above, whereas the second uses the same proof used to prove (36). As we get similar expansions for all $\underline{j} \in \mathcal{D}_5$ and $|\mathcal{D}_5| = O(n^5)$ we have

$$\frac{1}{n^6} \sum_{\underline{j} \in \mathcal{D}_5} \sum_{\pi_1 \in \mathcal{P}_1} A_{2,2,2}^j(\pi_1) = O\left(\frac{1}{\lambda n} + \frac{\log^2(a)}{n\lambda^2}\right).$$

Similarly we can show that

$$\frac{1}{n^6} \sum_{\underline{j} \in \mathcal{D}_4} \sum_{\pi_1 \in \mathcal{P}_1} A_{2,2,2}^j(\pi_1) = O\left(\frac{1}{\lambda n^2} + \frac{\log^2(a)}{n^2\lambda^2}\right).$$

and

$$\frac{1}{n^6} \sum_{\underline{j} \in \mathcal{D}_3} \sum_{\pi_1 \in \mathcal{P}_1} A_{2,2,2}^j(\pi_1) = O\left(\frac{1}{n^3} + \frac{\log^2(a)}{n^3\lambda^2}\right).$$

Therefore, if $n \gg \lambda / \log^2(a)$ we have $B_{2,2,2} = O(\frac{\log^2(a)}{\lambda^2})$.

Bound for $B_{4,2}$

To bound $B_{4,2}$ we consider the ‘typical’ partition given in (40). Since $A_{4,2}^j(\pi_{2,1})$ involves fourth order cumulants by Lemma A.7(ii) it will be zero in the case that the \underline{j} are all different. Therefore, only a maximum of five terms in \underline{j} can be different, which gives

$$\frac{1}{n^6} \sum_{\underline{j} \in \mathcal{D}} A_{4,2}^j(\pi_{2,1}) = \frac{1}{n^6} \sum_{i=1}^3 \sum_{\underline{j} \in \mathcal{D}_{6-i}} A_{4,2}^j(\pi_{2,1}).$$

We will show that for $\underline{j} \in \mathcal{D}_5$, $A_{2,4}^j(\pi_{2,1})$ will not be as small as $O(\log^2(a)/\lambda^2)$, however, this will be compensated by $|\mathcal{D}_5| = O(n^5)$. Let $\underline{j} = (j_1, j_2, j_3, j_4, j_1, j_6)$, then expanding the fourth order cumulant in $A_{4,2}^j(\pi_{2,1})$ and using conditional cumulants (see (33)) we have

$$\begin{aligned}
& A_{4,2}^j(\pi_{2,1}) \\
&= \sum_{k_1, k_2, k_3=-a}^a \left\{ \text{cov}[c(s_1 - s_2)e^{i\omega_{k_1}(s_1-s_2)}, c(s_3 - s_1)e^{is_3\omega_{k_2}-is_1\omega_{k_3}}] \right. \\
&\quad + \text{cov}[c(s_1 - s_3)e^{i(s_1\omega_{k_1}+s_3\omega_{k_2})}, c(s_1 - s_2)e^{is_1\omega_{k_3}-is_2\omega_{k_1}}] \\
&\quad \left. + \text{cov}[c(0)e^{is_1(\omega_{k_1}+\omega_{k_3})}, c(s_2 - s_3)e^{-is_2\omega_{k_1}+is_3\omega_{k_2}}] \right\} E[c(s_4 - s_6)e^{-is_4\omega_{k_2}-is_6\omega_{k_3}}] \\
&= A_{42}^j(\pi_{2,1}, \Omega_1) + A_{42}^j(\pi_{2,1}, \Omega_2) + A_{42}^j(\pi_{2,1}, \Omega_3), \tag{47}
\end{aligned}$$

where we use the notation Ω to denote the partition of the fourth order cumulant into it's conditional cumulants. To bound each term we expand the covariances as expectations, this gives

$$\begin{aligned}
& A_{42}^j(\pi_{2,1}, \Omega_1) \\
&= \frac{1}{n^6} \sum_{\underline{j} \in \mathcal{D}_5} \sum_{k_1, k_2, k_3=-a}^a \left\{ E[c(s_1 - s_2)c(s_3 - s_1)e^{i\omega_{k_1}(s_1-s_2)}e^{is_3\omega_{k_2}-is_1\omega_{k_3}}] E[c(s_4 - s_6)e^{-is_4\omega_{k_2}-is_6\omega_{k_3}}] - \right. \\
&\quad \left. E[c(s_1 - s_2)e^{i\omega_{k_1}(s_1-s_2)}] E[c(s_3 - s_1)e^{is_3\omega_{k_2}-is_1\omega_{k_3}}] E[c(s_4 - s_6)e^{-is_4\omega_{k_2}-is_6\omega_{k_3}}] \right\} \\
&= A_{42}^j(\pi_{2,1}, \Omega_1, \Pi_1) + A_{42}^j(\pi_{2,1}, \Omega_1, \Pi_2),
\end{aligned}$$

where we use the notation Π to denote the expansion of the cumulants of the spatial covariances expanded into expectations. To bound $A_{42}^j(\pi_{2,1}, \Omega_1, \Pi_1)$ we use the spectral representation theorem to give

$$\begin{aligned}
& A_{42}^j(\pi_{2,1}, \Omega_1, \Pi_1) = \\
& \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \sum_{k_1, k_2, k_3=-a}^a f(x)f(y)f(z) \text{sinc}\left(\frac{\lambda(x+y)}{2} + (k_1 - k_3)\pi\right) \text{sinc}\left(\frac{\lambda x}{2} + k_1\pi\right) \times \\
& \text{sinc}\left(\frac{\lambda y}{2} + k_2\pi\right) \text{sinc}\left(\frac{\lambda z}{2} - k_2\pi\right) \text{sinc}\left(\frac{\lambda z}{2} + k_3\pi\right) dx dy dz.
\end{aligned}$$

By changing variables

$$\begin{aligned}
A_{42}^j(\pi_{2,1}, \Omega_1, \Pi_1) &= \frac{1}{\pi^3 \lambda^3} \sum_{k_1, k_2, k_3=-a}^a \int_{\mathbb{R}^3} f\left(\frac{2u}{\lambda} - \omega_{k_1}\right) f\left(\frac{2v}{\lambda} - \omega_{k_2}\right) f\left(\frac{2w}{\lambda} - \omega_{k_3}\right) \times \\
& \text{sinc}(u + v + (k_2 + k_3)\pi) \text{sinc}(u) \text{sinc}(v) \text{sinc}(w) \text{sinc}(w - (k_2 + k_3)\pi) du dv dw.
\end{aligned}$$

Just as in the bound for $B_{2,2,2}$, we represent the k s inside the sinc function as a set of linear equations

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}, \quad (48)$$

which has rank one. We make a change of variables $m = k_2 + k_3$, $k_1 = k_1$ and $k_2 = k_2$ to give

$$\begin{aligned} & A_{42}^j(\pi_{2,1}, \Omega_1, \Pi_1) \\ &= \frac{1}{\pi^3 \lambda} \int_{\mathbb{R}^3} \frac{1}{\lambda^2} \sum_{k_1, k_2 = -a}^a f\left(\frac{2u}{\lambda} - \omega_{k_1}\right) f\left(\frac{2v}{\lambda} - \omega_{k_2}\right) f\left(\frac{2w}{\lambda} - \omega_{m_1 - k_2}\right) \times \\ & \quad \sum_m \text{sinc}(u + v + m\pi) \text{sinc}(u) \text{sinc}(v) \text{sinc}(w) \text{sinc}(w - m\pi) du dv dw \\ &= \frac{2^3}{\lambda} \int_{\mathbb{R}^3} \sum_{m_1} G_{\lambda, m}\left(\frac{2u}{\lambda}, \frac{2v}{\lambda}, \frac{2w}{\lambda}\right) \text{sinc}(u + v + m\pi) \text{sinc}(u) \text{sinc}(v) \text{sinc}(w) \text{sinc}(w - m\pi) du dv dw, \end{aligned}$$

where $G_{\lambda, m}\left(\frac{2u}{\lambda}, \frac{2v}{\lambda}, \frac{2w}{\lambda}\right) = \frac{1}{\lambda^2} \sum_{k_1, k_2 = -a}^a f\left(\frac{2u}{\lambda} - \omega_{k_1}\right) f\left(\frac{2v}{\lambda} - \omega_{k_2}\right) f\left(\frac{2w}{\lambda} - \omega_{m - k_2}\right)$. Taking absolutes gives

$$|A_{42}^j(\pi_{2,1}, \Omega_1, \Pi_1)| \leq \frac{C}{\lambda} \int_{\mathbb{R}^3} \sum_m |\text{sinc}(u + v + m\pi) \text{sinc}(w - m\pi) \text{sinc}(u) \text{sinc}(v) \text{sinc}(w)| du dv dw$$

Since the above contains m in the sinc function we use Lemma A.1(i) to show

$$\begin{aligned} |A_{42}^j(\pi_{2,1}, \Omega_1, \Pi_1)| &\leq C \int_{\mathbb{R}^3} \sum_m |\text{sinc}(u + v + m\pi) \text{sinc}(w - m\pi) \text{sinc}(u) \text{sinc}(v) \text{sinc}(w)| du dv dw \\ &\leq C \sum_m \ell_0(m\pi) \ell_1(m\pi) = O(1), \end{aligned}$$

where the functions $\ell_0(\cdot)$ and $\ell_1(\cdot)$ are defined in Lemma A.1(i). Thus $|A_{42}^j(\pi_{2,1}, \Omega_1, \Pi_1)| = O(\frac{1}{\lambda})$. We use the same method used to bound (46) to show that $A_{42}^j(\pi_{2,1}, \Omega_1, \Pi_2) = O(\frac{\log^2(a)}{\lambda^2})$ and $|A_{42}^j(\pi_{2,1}, \Omega_2)| = O(\frac{1}{\lambda})$. Furthermore, it is straightforward to see that by the independence of s_1 and s_2 and s_3 that $A_{42}^j(\pi_{2,1}, \Omega_2) = 0$ (recalling that $A_{42}^j(\pi_{2,1}, \Omega_2)$ is defined in equation (47)). Thus altogether we have for $\underline{j} = (j_1, j_2, j_3, j_4, j_1, j_6)$ and this partition $\pi_{2,1}$, that $|A_{42}^j(\pi_{2,1})| = O(\frac{1}{\lambda})$. However, it is important to note for all other $\underline{j} \in \mathcal{D}_5$ and partitions in \mathcal{P}_2 the same method will lead to a similar decomposition given in (47) and the rank one matrix given in (48). The rank one matrix means one ‘free’ m in the sinc functions and this $|A_{42}^j(\pi_2)| = O(\frac{1}{\lambda})$ for all $\underline{j} \in \mathcal{D}_5$ and $\pi_2 \in \mathcal{P}_2$. Thus, since $|\mathcal{D}_5| = O(n^5)$ we have

$$\frac{1}{n^6} \sum_{\underline{j} \in \mathcal{D}_5} \sum_{\pi_2 \in \mathcal{P}_{42}} A_{4,2}^j(\pi_2) = O\left(\frac{1}{\lambda n} + \frac{\log^2(a)}{\lambda^2 n}\right) = O\left(\frac{\log^2(a)}{\lambda^2}\right)$$

if $n \gg \lambda / \log^2(a)$. For $\underline{j} \in \mathcal{D}_4$ and $\underline{j} \in \mathcal{D}_3$ we use the same argument, noting that the number of free m 's in the sinc functions goes down but to compensate, $|\mathcal{D}_4| = O(n^4)$ and $|\mathcal{D}_3| = O(n^3)$. Therefore

$$B_{4,2} = \frac{1}{n^6} \sum_{i=1}^3 \sum_{\underline{j} \in \mathcal{D}_{6-i}} \sum_{\pi_2 \in \mathcal{P}_2} A_{4,2}^{\underline{j}}(\pi_2) = O\left(\frac{\log^2(a)}{\lambda^2}\right),$$

if $n \gg \log^2(a)/\lambda$.

Bound for B_6

Finally, we bound B_6 . By using Lemma A.7(ii) we observe that $A_6^{\underline{j}}(k_1, k_2, k_3) = 0$ if more than four elements of \underline{j} are different. Thus

$$B_6 = \frac{1}{n^6} \sum_{i=2}^3 \sum_{\underline{j} \in \mathcal{D}_{6-i}} A_6^{\underline{j}}.$$

We start by considering the case that $\underline{j} = (j_1, j_2, j_1, j_4, j_1, j_6)$ (three elements in \underline{j} are the same), then by using conditional cumulants we have

$$\begin{aligned} & A_6^{\underline{j}} \\ &= \sum_{k_1, k_2, k_3 = -a}^a \text{cum}[Z(s_1)e^{is_1\omega_{k_1}}, Z(s_2)e^{-is_2\omega_{k_1}}, Z(s_1)e^{is_1\omega_{k_2}}, Z(s_4)e^{-is_4\omega_{k_2}}, Z(s_1)e^{is_1\omega_{k_3}}, Z(s_6)e^{-is_6\omega_{k_3}}] \\ &= \sum_{k_1, k_2, k_3 = -a}^a \sum_{\Omega \in \mathcal{R}} A_6^{\underline{j}}(\Omega), \end{aligned}$$

where \mathcal{R} is the set of all pairwise partitions of $\{1, 2, 1, 4, 1, 6\}$, for example

$$A_6^{\underline{j}}(\Omega_1) = \text{cum}[c(s_1 - s_2)e^{i\omega_{k_1}(s_1 - s_2)}, c(s_1 - s_4)e^{i\omega_{k_2}(s_1 - s_4)}, c(s_1 - s_6)e^{i\omega_{k_3}(s_1 - s_6)}].$$

We will first bound the above and then explain how this generalises to the other $\Omega \in \mathcal{R}$ and $\underline{j} \in \mathcal{D}_4$. Expanding the above third order cumulant in terms of expectations gives

$$\begin{aligned}
& A_6^j(\Omega_1) \\
&= \sum_{k_1, k_2, k_3 = -a}^a \left\{ \mathbb{E}[c(s_1 - s_2)e^{i\omega_{k_1}(s_1-s_2)}c(s_1 - s_4)e^{i\omega_{k_2}(s_1-s_4)}c(s_1 - s_6)e^{i\omega_{k_3}(s_1-s_6)}] - \right. \\
&\quad \mathbb{E}[c(s_1 - s_2)e^{i\omega_{k_1}(s_1-s_2)}]\mathbb{E}[c(s_1 - s_4)e^{i\omega_{k_2}(s_1-s_4)}c(s_1 - s_6)e^{i\omega_{k_3}(s_1-s_6)}] - \\
&\quad \mathbb{E}[c(s_1 - s_2)e^{i\omega_{k_1}(s_1-s_2)}c(s_1 - s_4)e^{i\omega_{k_2}(s_1-s_4)}]\mathbb{E}[c(s_1 - s_6)e^{i\omega_{k_3}(s_1-s_6)}] - \\
&\quad \mathbb{E}[c(s_1 - s_2)e^{i\omega_{k_1}(s_1-s_2)}c(s_1 - s_6)e^{i\omega_{k_3}(s_1-s_6)}]\mathbb{E}[c(s_1 - s_4)e^{i\omega_{k_2}(s_1-s_4)}] + \\
&\quad \left. 2\mathbb{E}[c(s_1 - s_2)e^{i\omega_{k_1}(s_1-s_2)}]\mathbb{E}[c(s_1 - s_4)e^{i\omega_{k_2}(s_1-s_4)}]\mathbb{E}[c(s_1 - s_6)e^{i\omega_{k_3}(s_1-s_6)}] \right\} \\
&= \sum_{i=1}^5 A_6^j(\Omega_1, \Pi_i).
\end{aligned}$$

We observe that for $2 \leq k \leq 5$, $A_6^j(\Omega_1, \Pi_i)$ resembles $A_{4,2}^j(\pi_{2,1}, \Omega_1, \Pi_1)$ defined in (48), thus the same proof used to bound the terms in (48) can be use to show that $A_6^j(\Omega_1, \Pi_i) = O(\frac{1}{\lambda})$. However, the first term $A_6^j(\Omega_1, \Pi_1)$ involves just one expectation, and is not included in the previous cases. By using the spectral representation theorem we have

$$\begin{aligned}
& A_6(\Omega_1, \Pi_1) \\
&= \sum_{k_1, k_2, k_3 = -a}^a \mathbb{E}[c(s_1 - s_2)e^{i\omega_{k_1}(s_1-s_2)}c(s_1 - s_4)e^{i\omega_{k_2}(s_1-s_4)}c(s_1 - s_6)e^{i\omega_{k_3}(s_1-s_6)}] \\
&= \frac{1}{(2\pi)^3} \sum_{k_1, k_2, k_3 = -a}^a \int \int \int f(x)f(y)f(z) \text{sinc}\left(\frac{\lambda(x+y+z)}{2} + (k_1 + k_2 + k_3)\pi\right) \text{sinc}\left(\frac{\lambda x}{2} + k_1\pi\right) \\
&\quad \text{sinc}\left(\frac{\lambda y}{2} + k_2\pi\right) \text{sinc}\left(\frac{\lambda z}{2} + k_3\pi\right) dx dy dz \\
&= \frac{2^3}{(2\pi)^3 \lambda^3} \sum_{k_1, k_2, k_3 = -a}^a \int \int \int f\left(\frac{2u}{\lambda} - \omega_{k_1}\right) f\left(\frac{2v}{\lambda} - \omega_{k_2}\right) f\left(\frac{2w}{\lambda} - \omega_{k_3}\right) \times \\
&\quad \text{sinc}(u+v+w) \text{sinc}(u) \text{sinc}(v) \text{sinc}(w) du dv dw.
\end{aligned}$$

It is obvious that the k s within in the sinc function correspond to a rank zero matrix, and thus $A_6(\Omega_1, \Pi_1) = O(1)$. Therefore, $A_6^j(\Omega_1) = O(1)$. A similar bound holds for all $\underline{j} \in \mathcal{D}_4$, this we have

$$\frac{1}{n^6} \sum_{\underline{j} \in \mathcal{D}_4} A_6^j(\Omega_1) = O\left(\frac{1}{n^2}\right),$$

since $|\mathcal{D}_4| = O(n^4)$. Indeed, the same argument applies to the other partitions Ω and $\underline{j} \in \mathcal{D}_3$, thus altogether we have

$$\mathcal{B}_6 = \frac{1}{n^6} \sum_{\Omega \in \mathcal{R}} \sum_{i=2}^3 \sum_{\underline{j} \in \mathcal{D}_{6-i}} A_6^{\underline{j}}(\Omega) = O\left(\frac{1}{n^2}\right).$$

Altogether, using the bounds derived for $\mathcal{B}_{2,2,2}$, $\mathcal{B}_{4,2}$ and \mathcal{B}_6 we have

$$\text{cum}_4(\tilde{Q}_{a,\lambda}(1, 0)) = O\left(\frac{\log^2(a)}{\lambda^2} + \frac{1}{n\lambda} + \frac{\log^2(a)}{\lambda^2 n} + \frac{1}{n^2}\right) = O\left(\frac{\log^2(a)}{\lambda^2}\right),$$

where the last bound is due to the conditions on a, n and λ . This gives the result. \square

We now generalize the above results to higher order cumulants.

Lemma A.9 *Suppose Assumptions 2.1, 2.2 and 2.4(i)(a,b) or (ii)(a,b) are satisfied, and $d = 1$. Then we have*

$$\text{cum}_q[\tilde{Q}_{a,\lambda}(g, r)] = O\left(\frac{\log^{2(q-2)}(a)}{\lambda^{q-1}}\right), \quad (49)$$

$$\text{cum}_q[\tilde{Q}_{a,\lambda}(g, r_1), \dots, \tilde{Q}_{a,\lambda}(g, r_q)] = O\left(\frac{\log^{2(q-1)}(a)}{\lambda^{q-1}}\right) \quad (50)$$

and in the case $d > 1$, we have

$$\text{cum}_q[\tilde{Q}_{a,\lambda}(g, \mathbf{r}_1), \dots, \tilde{Q}_{a,\lambda}(g, \mathbf{r}_q)] = O\left(\frac{\log^{2d(q-1)}(a)}{\lambda^{d(q-1)}}\right) \quad (51)$$

with $\frac{\lambda}{n \log^2(a)} \rightarrow 0$, $\log^2(a)/\lambda \rightarrow \infty$ as $\lambda \rightarrow \infty$, $n \rightarrow \infty$ and $a \rightarrow \infty$.

PROOF. The proof essentially follows the same used to bound the second and third cumulants. We first prove (49). To simplify the notation we prove the result for $g = 1$ and $r = 0$, noting that the proof in the general case is identical. Expanding out $\text{cum}_q[\tilde{Q}_{a,\lambda}(1, 0)]$ using indecomposable partitions gives

$$\begin{aligned} & \text{cum}_q[\tilde{Q}_{a,\lambda}(1, 0)] \\ &= \frac{1}{n^{2q}} \sum_{j_1, \dots, j_{2q} \in \mathcal{D}} \sum_{k_1, \dots, k_q = -a}^a \text{cum} \left[Z(s_{j_1}) Z(s_{j_2}) e^{i\omega_{k_1}(s_{j_1} - s_{j_2})}, \dots, Z(s_{j_{2q-1}}) Z(s_{j_{2q}}) e^{i\omega_{k_q}(s_{j_{2q-1}} - s_{j_{2q}})} \right] \\ &= \frac{1}{n^{2q}} \sum_{\underline{j} \in \mathcal{D}} \sum_{\underline{b} \in \mathcal{B}_q} \sum_{\pi_{\underline{b}} \in \mathcal{P}_{\underline{b}}} A_{2\underline{b}}^{\underline{j}}(\pi_{\underline{b}}) \end{aligned}$$

where \mathcal{B}_q corresponds to the set of integer partitions of q (more precisely, each partition is a sequence of positive integers which sums to q). Let $\underline{b} = (b_1, \dots, b_m)$ denote one of these partitions, then $\mathcal{P}_{\underline{b}}$ is the set of all indecomposable partitions of $\{(1, 2), (3, 4), \dots, (2q-1, 2q)\}$ where the size of each partition is $2b_1, 2b_2, \dots, 2b_m$. For example, if $q = 3$, then one example of an element of \mathcal{B}_3 is $\underline{b} = (1, 1, 1)$ and $\mathcal{P}_{(1,1,1)}$ corresponds to all pairwise indecomposable partitions of $\{(1, 2), (3, 4), (5, 6)\}$. Finally, $A_{2\underline{b}}^j(\pi_{\underline{b}})$ corresponds to the indecomposable partition of the cumulant $\text{cum}[Z(s_{j_1})Z(s_{j_2})e^{i\omega_{k_1}(s_{j_1}-s_{j_2})}, \dots, Z(s_{j_{2q-1}})Z(s_{j_{2q}})e^{i\omega_{k_q}(s_{j_{2q-1}}-s_{j_{2q}})}]$, where the product of the cumulants are of order $2b_1, 2b_2, \dots, 2b_m$ (examples, in the case $q = 3$ are given in equation (39)-(41)). Let

$$B_{2\underline{b}} = \frac{1}{n^{2q}} \sum_{\underline{j} \in \mathcal{D}} \sum_{\pi_{\underline{b}} \in \mathcal{P}_{\underline{b}}} A_{2\underline{b}}^j(\pi_{\underline{b}}),$$

therefore $\text{cum}_q[\tilde{Q}_{a,\lambda}(1, 0)] = \sum_{\underline{b} \in \mathcal{B}_q} B_{2\underline{b}}$.

Just as in the proof of Lemma A.8, we will show that under the condition $n \gg \lambda/\log^2(a)$, the pairwise decomposition $B_{2,\dots,2}$ is the denominating term. We start with a ‘typical’ decomposition $\pi_{(2,\dots,2),1} \in \mathcal{P}_{2,\dots,2}$,

$$A_{2,\dots,2}^j(\pi_{(2,\dots,2),1}) = \sum_{k_1, \dots, k_q = -a}^a \text{cov}[Z(s_1)e^{is_1\omega_{k_1}}, Z(s_{2q})e^{-is_{2q}\omega_{k_q}}] \prod_{c=1}^{q-1} \text{cov}[Z(s_{2c})e^{-is_{2c}\omega_{k_c}}, Z(s_{2c+1})e^{is_{2c+1}\omega_{k_{c+1}}}]$$

and

$$\frac{1}{n^{2q}} \sum_{\underline{j} \in \mathcal{D}} A_{2,\dots,2}^j(\pi_{(2,\dots,2),1}) = \frac{1}{n^{2q}} \sum_{i=0}^{q-1} \sum_{\underline{j} \in \mathcal{D}_{2q-i}} A_{2,\dots,2}^j(\pi_{(2,\dots,2),1}),$$

where \mathcal{D}_{2q} denotes the set where all elements of \underline{j} are different and \mathcal{D}_{2q-i} denotes the set that $(2q-i)$ elements in \underline{j} are different. We first consider the case that $\underline{j} = (1, 2, \dots, 2q) \in \mathcal{D}_{2q}$. Using identical arguments to those used for the second and third order cumulants we can show that

$$\begin{aligned} A_{2,\dots,2}^j(\pi_{(2,\dots,2),1}) &= \sum_{k_1, \dots, k_q = -a}^a \int_{\mathbb{R}^q} f(x_q) \text{sinc}\left(\frac{2x_q}{\lambda} + k_1\pi\right) \text{sinc}\left(\frac{2x_q}{\lambda} + k_q\pi\right) \times \\ &\quad \prod_{c=1}^{q-1} f(x_c) \text{sinc}\left(\frac{2x_c}{\lambda} - k_c\pi\right) \text{sinc}\left(\frac{2x_c}{\lambda} + k_{c+1}\pi\right) \prod_{c=1}^q dx_c. \end{aligned} \quad (52)$$

By a change of variables we get

$$\begin{aligned} A_{2,\dots,2}^j(\pi_{(2,\dots,2),1}) &= \frac{1}{\lambda^q} \sum_{k_1, \dots, k_q = -a}^a \int_{\mathbb{R}^q} \prod_{c=1}^{q-1} f\left(\frac{\lambda u_c}{2} + \omega_c\right) \text{sinc}(u_c) \text{sinc}(u_c + (k_{c+1} - k_c)\pi) \\ &\quad \times f\left(\frac{\lambda u_q}{2} + \omega_1\right) \text{sinc}(u_q) \text{sinc}(u_q + (k_q - k_1)\pi) \prod_{c=1}^q dx_c. \end{aligned}$$

As in the proof of the third order cumulant we can rewrite the k s in the above as a matrix equation

$$\begin{pmatrix} -1 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & 0 \\ -1 & 0 & \dots & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_q \end{pmatrix},$$

noting that that above is a $(q-1)$ -rank matrix. Therefore applying the same arguments that were used in the proof of the third order cumulant and also Lemma A.1(iii) we can show that the $A_{2,\dots,2}^j(\pi_{(2,\dots,2),1}) = O(\frac{\log^{2(q-2)}(a)}{\lambda^{q-1}})$. Thus for $\underline{j} \in \mathcal{D}_{2q}$ we have $\frac{1}{n^6} \sum_{\underline{j} \in \mathcal{D}_{2q}} A_{2,\dots,2}^j(\pi_{(2,\dots,2),1}) = O(\frac{\log^{2(q-2)}(a)}{\lambda^{q-1}})$.

In the case that $\underline{j} \in \mathcal{D}_{2q-1}$ ($(2q-1)$ -terms in \underline{j} are different) by using the same arguments as those used to bound $A_{2,2,2}$ (in the third order cumulant case) we have $A_{2,\dots,2}^j(\pi_{(2,\dots,2),1}) = O(\frac{\log^{2(q-3)}(a)}{\lambda^{q-2}})$, similarly if $(2q-2)$ -terms in \underline{j} are different, then $A_{2,\dots,2}^j(\pi_{(2,\dots,2),1}) = O(\frac{\log^{2(q-4)}(a)}{\lambda^{q-3}})$ and so forth. Therefore, since $|\mathcal{D}_{2q-i}| = O(n^{2q-i})$ we have

$$\frac{1}{n^{2q}} \sum_{\underline{j} \in \mathcal{D}} |A_{2,\dots,2}^j(\pi_{(2,\dots,2),1})| \leq C \sum_{i=0}^q \frac{\log^{2(q-2-i)}(a)}{\lambda^{q-1-i} n^i}.$$

Now by using that $n \gg \lambda / \log^2(a)$ we have

$$\frac{1}{n^{2q}} \sum_{\underline{j} \in \mathcal{D}} |A_{2,\dots,2}^j(\pi_{(2,\dots,2),1})| = O\left(\frac{\log^{2(q-2)}(a)}{\lambda^{q-1}}\right).$$

The same argument holds for every other covariance indecomposable partition because the corresponding matrix will always have rank $(q-1)$ (see Deo and Chen (2000)), thus $B_{2,\dots,2} = O(\frac{\log^{2(q-1)}(a)}{\lambda^{q-1}})$.

Now, we bound the other extreme B_{2q} . Using the conditional cumulant expansion (33) and noting that $\text{cum}_{2q}[Z(s_{j_1})e^{is_{j_1}\omega_{k_1}}, \dots, Z(s_{j_{2q}})e^{-is_{j_{2q}}\omega_{k_q}}]$ is non-zero, only when at most $(q+1)$ elements of \underline{j} are different we have

$$\begin{aligned} B_{2q} &= \frac{1}{n^{2q}} \sum_{\underline{j} \in \mathcal{D}} \sum_{k_1, \dots, k_q = -a}^a \text{cum}_{2q}[Z(s_{j_1})e^{is_{j_1}\omega_{k_1}}, \dots, Z(s_{j_{2q}})e^{-is_{j_{2q}}\omega_{k_q}}] \\ &= \frac{1}{n^{2q}} \sum_{\underline{j} \in \mathcal{D}_{q+1}} \sum_{\Omega \in \mathcal{R}_{2q}} A_{2q}^j(\Omega). \end{aligned}$$

where \mathcal{R}_{2q} is the set of all pairwise partitions of $\{1, \dots, 2q\}$. We consider a ‘typical’ partition

$$\Omega_1 \in \mathcal{R}_{2q}$$

$$A_{2q}^j(\Omega_1) = \sum_{k_1, \dots, k_q = -a}^a \text{cum}[c(s_1 - s_2)e^{i(s_1 - s_2)\omega_{k_1}}, \dots, c(s_{2q-1} - s_{2q})e^{i(s_{2q-1} - s_{2q})\omega_{k_q}}]. \quad (53)$$

By expanding the above the cumulant as the sum of the product of expectations we have

$$A_{2q}^j(\Omega_1) = \frac{1}{n^{2q}} \sum_{\underline{j} \in \mathcal{D}_{q+1}} \sum_{\Omega \in \mathcal{R}_{2q}} \sum_{\Pi \in \mathcal{S}_q} A_{2q}^j(\Omega_1, \Pi),$$

where \mathcal{S}_q is the set of all partitions of $\{1, \dots, q\}$. As we have seen in both the second and order cumulant calculations, the leading term in the cumulant expansion is the expectation over all the covariance terms. The same result holds true for higher order cumulants, the expectation over all the covariances in that cumulant is the leading term because the it gives the linear equation of the k s in the sinc function with the lowest order rank (we recall the lower the rank the less ‘free’ λ s). Based on this we will only derive bounds for the expectation over all the covariances. Let $\Pi_1 \in \mathcal{S}_q$, where

$$A_{2q}^j(\Omega_1, \Pi_1) = \sum_{k_1, \dots, k_q = -a}^a \mathbb{E}\left[\prod_{c=1}^q c(s_{2q-1} - s_{2q})e^{i(s_{2q-1} - s_{2q})\omega_{k_q}}\right].$$

Representing the above expectation as an integral and using the spectral representation theorem and a change of variables gives

$$\begin{aligned} A_{2q}^j(\Omega_1, \Pi_1) &= \sum_{k_1, \dots, k_q = -a}^a \mathbb{E}\left[\prod_{c=1}^q c(s_{2q-1} - s_{2q})e^{i(s_{2q-1} - s_{2q})\omega_{k_q}}\right] \\ &= \frac{1}{(2\pi)^q} \sum_{k_1, k_2, k_3 = -a}^a \int_{\mathbb{R}^q} \text{sinc}\left(\frac{\lambda(\sum_{c=1}^q x_c)}{2} + \pi \sum_{c=1}^q k_c\right) \prod_{c=1}^q f(x_c) \text{sinc}(x_c + k_c \pi) \prod_{c=1}^q dx_c \\ &= \frac{2^q}{(2\pi)^q \lambda^q} \sum_{k_1, \dots, k_q = -a}^a \int_{\mathbb{R}^q} \text{sinc}\left(\sum_{c=1}^q u_c\right) \prod_{c=1}^q f\left(\frac{2u_c}{\lambda} - \omega_{k_c}\right) \text{sinc}(u_c) du_c = O(1), \end{aligned}$$

where the last line follows from Lemma A.1, equation (28). Therefore, $A_{2q}^j(\Omega_1) = O(1)$. By using the same method on every partition $\Omega \in \mathcal{R}_{q+1}$ and $\underline{j} \in \mathcal{D}_{q+1}$ and $|\mathcal{D}_{q+1}| = O(n^{q+1})$, we have

$$B_{2q} = \frac{1}{n^{2q}} \sum_{\underline{j} \in \mathcal{D}_{q+1}} \sum_{\Omega \in \mathcal{R}_{2q}} \sum_{\Pi \in \mathcal{S}_{2q}} A_{2q}^j(\Omega_1, \Pi_1) = O\left(\frac{1}{n^{q-1}}\right).$$

Finally, we briefly discuss the terms $B_{2\underline{b}}$ which lie between the two extremes $B_{2, \dots, 2}$ and B_{2q} . Since $B_{2\underline{b}}$ is the product of $2b_1, \dots, 2b_m$ cumulants, by Lemma A.7(ii) at most $\sum_{j=1}^m (b_j +$

1) = $q + m$ elements of \underline{j} can be different. Thus

$$B_{2\underline{b}} = \frac{1}{n^{2q}} \sum_{i=q}^{q+m} \sum_{\underline{j} \in \mathcal{D}_i} \sum_{\pi_{\underline{b}} \in \mathcal{P}_{\underline{b}}} A_{2\underline{b}}^j(\pi_{\underline{b}}).$$

By expanding the cumulants in terms of the cumulants of covariances conditioned on the location (which is due to Gaussianity of the random field, see for example, (53)) we have

$$B_{2\underline{b}} = \frac{1}{n^{2q}} \sum_{i=q}^{q+m} \sum_{\underline{j} \in \mathcal{D}_i} \sum_{\pi_{\underline{b}} \in \mathcal{P}_{\underline{b}}} \sum_{\Omega \in \mathcal{R}_{2\underline{b}}} A_{2\underline{b}}^j(\pi_{\underline{b}}, \Omega),$$

where $\mathcal{R}_{2\underline{b}}$ is the set of all paired partitions of $\{(1, \dots, 2b_1), \dots, (2b_{m-1} + 1, \dots, 2b_m)\}$. The leading terms are the highest order expectations which leads to a matrix equation for the k 's within the sinc function which has rank at least $(m - 1)$ (we do not give a formal proof of this). Therefore, $B_{2\underline{b}} = O(\frac{\log^{2(m-1)}(a)}{n^{q-m}\lambda^{m-1}}) = O(\frac{\log^{2(q-2)}(a)}{\lambda^{q-1}})$ (since $n \gg \lambda/\log^2(a)$). This concludes the proof of (49).

The proof of (50) is identical and we omit the details.

To prove the result for $d > 1$, (51) we use the same method, the main difference is that the spectral density function in (52) is a multivariate function of dimension d , there are $2dp$ sinc functions and the integral is over \mathbb{R}^{dp} , however the analysis is identical. \square \square

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